Disasters Implied by Equity Index Options

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November 18, 2009

Summary

- Problem: disasters infrequent $\Rightarrow$ hard to estimate their distribution
- Solution: infer from option prices (market prices of bets on disasters)
- What we find
  - disasters apparent in options data
  - more modest than disasters in macro data

Outline

- Preliminaries: cumulants, entropy, AJ bound
- Three objects of interest in financial economics: true probabilities $p(x)$, risk-neutral probabilities $p^*(x)$, stochastic discount factor $m(x)$

$$
\begin{aligned}
p^*(x) &= R_f \cdot p(x) m(x) \\
m(x) &= \frac{1}{R_f} \cdot \frac{p^*(x)}{p(x)}
\end{aligned}
$$

- Macro-finance: m and p
- Option-pricing: p and p^*
- Third possibility: m and p^*

Cumulants

- Cumulant generating function of a random variable x

$$k(s) \;=\; \log \mathbb{E} e^{sx} \;=\; \sum_{j=1}^{\infty} \kappa_j(x) s^j / j!$$

- Cumulants are closely related to moments

$$\begin{aligned}
\kappa_1 &= k'(0) &&= \text{mean} \\
\kappa_2 &= k''(0) &&= \text{variance, } \sigma^2 \\
\kappa_3 &= k'''(0) &&= \sigma^3 \times \text{ skewness} \\
\kappa_4 &= k''''(0) &&= \sigma^4 \times \text{ excess kurtosis}
\end{aligned}$$

- If x is normal, $\kappa_j = 0$ for $j > 2$

Entropy

- Entropy, $L(x)$, of a random variable $x > 0$ is $\log \mathbb{E}x - \mathbb{E} \log x$
- A measure of the variability of x
- Hans-Otto Georgii (quoted by Hansen and Sargent):

> *When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: "Call it entropy. It is already in use under that name and, besides, it will give you a great edge in debates because nobody knows what entropy is anyway."*

Alvarez-Jerman bound

- Entropy: for $x > 0$

$$L(x) \equiv \log \mathbb{E} x - \mathbb{E} \log x \geq 0$$

- AJ bound

$$L(m) \geq \mathbb{E} \left(\log r^j - \log r^1 \right)$$

Alvarez-Jerman bound

- Entropy: for $x > 0$

$$L(x) \equiv \log \mathbb{E}x - \mathbb{E} \log x \geq 0$$

- AJ bound

$$L(m) \geq \mathbb{E}\left(\log r^j - \log r^1\right)$$

Relates a measure of variability of the stochastic discount factor to a risk-adjusted measure of expected returns

Alvarez-Jerman bound vs. Hansen-Jagannathan bound

- Entropy: for $x > 0$

$$L(x) \equiv \log \mathbb{E}x - \mathbb{E} \log x \geq 0$$

- HJ: for $x > 0$

$$HJ(x) \equiv \frac{\sigma(x)}{\mathbb{E}x} \geq 0$$

- AJ bound

$$L(m) \geq \mathbb{E}\left(\log r^j - \log r^1\right)$$

- HJ bound

$$HJ(m) \geq \frac{\mathbb{E}r^j - r^1}{\sigma(r^j - r^1)}$$

Relates a measure of variability of the stochastic discount factor to a risk-adjusted measure of expected returns

Entropy and cumulants

- Entropy of pricing kernel

$$L(m) \;=\; \log \mathbb{E} e^{\log m} - \mathbb{E} \log m \;=\; \sum_{j=2}^{\infty} \kappa_j(\log m)/j!$$

- Zin's "never a dull moment" conjecture

$$L(m) \;=\; \kappa_2(\log m)/2! + \underbrace{\kappa_3(\log m)/3! + \kappa_4(\log m)/4! + \cdots}_{\text{high-order cumulants (incl disasters)}}$$

- In a lognormal model, all the higher cumulants are zero

Entropy and cumulants

- Can calculate contribution of odd cumulants and even cumulants separately—eg,

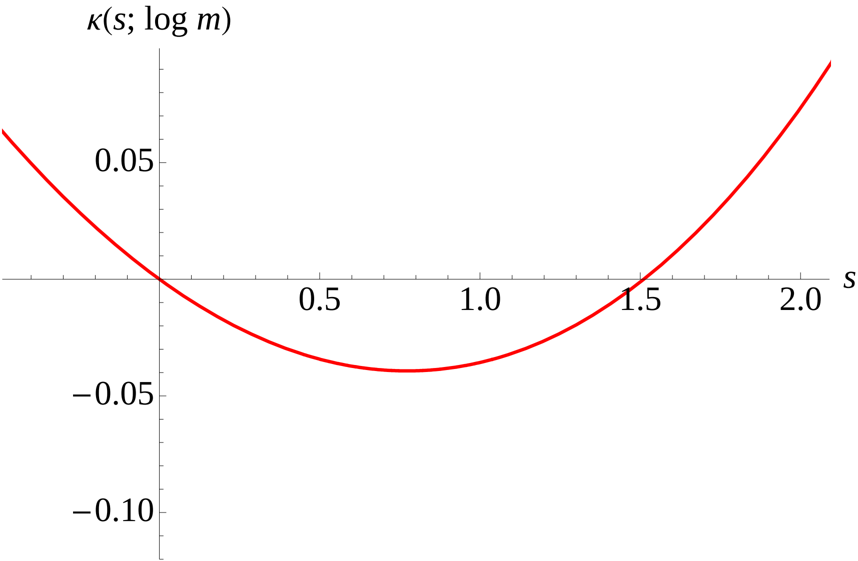
$$\sum_{j \text{ even}} \kappa_j(x) s^j / j! = [k(s) + k(-s)]/2$$

- Since $m(x) = R_f \cdot p^*(x)/p(x)$, we have $L(m) = L(p^*/p)$, and hence

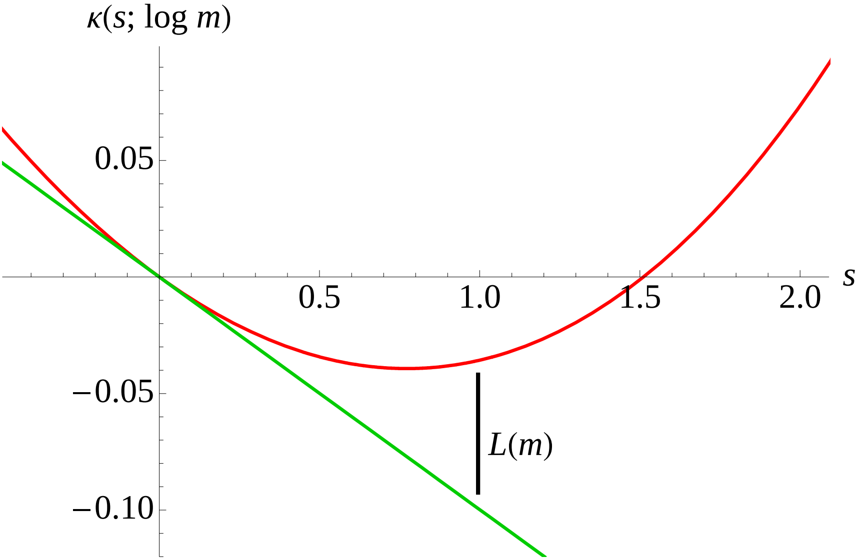
$$\begin{aligned} L(m) &= \log \mathbb{E} p^*/p - \mathbb{E} \log p^*/p \\ &= \log \sum_x p(x) \left[p^*(x)/p(x) \right] - \mathbb{E} \log p^*/p \\ &= -\mathbb{E} \log p^*/p \end{aligned}$$

(aka "relative entropy" or "Kullback-Leibler divergence")

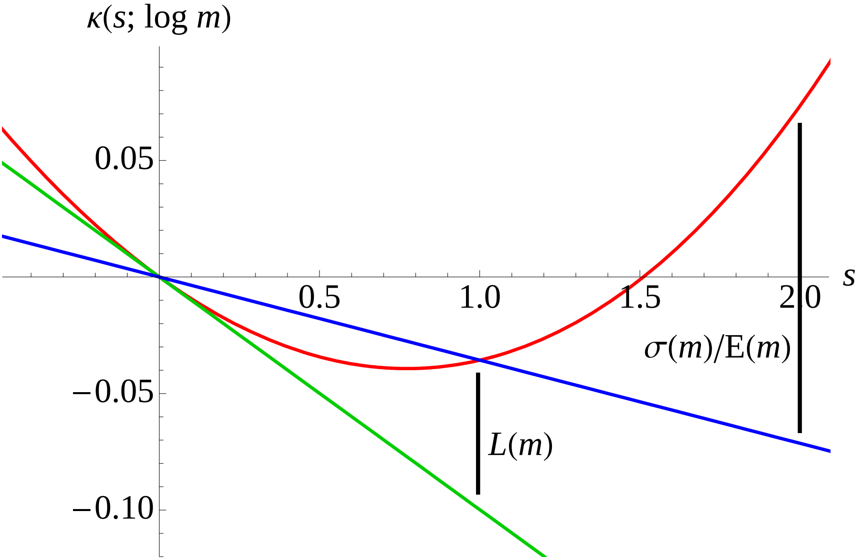
Entropy and cumulants



Entropy and cumulants



Entropy and cumulants



Plan of attack

- Modelling assumptions
  - i.i.d.
  - Tight link between consumption growth and equity returns
  - Representative agent with power utility [when needed]
- Parameter choices
  - Match mean and variance of log consumption growth
  - Ditto log equity return
  - Base "disasters" on macroeconomic evidence (Barro, Barro-Ursua)
  - Or on equity index options
- Compare macro- and option-based examples

Macro disasters: environment

- Consumption growth and "equity" return

$$
\begin{aligned}
g_{t+1} &= c_{t+1}/c_t \\
d_t &= c_t^{\lambda} \\
\log r^e_{t+1} &= \text{constant} + \lambda \log g_{t+1}
\end{aligned}
$$

- Power utility

$$\log m_{t+1} = \log \beta - \alpha \log g_{t+1}$$

- Yaron's "bazooka"

$$\kappa_j(\log m)/j! = \kappa_j(\log g)(-\alpha)^j/j!$$

Macro disasters: Poisson-normal mixture

- Consumption growth

$$\begin{aligned} \log g_{t+1} &= w_{t+1} + z_{t+1} \\ w_{t+1} &\sim \mathcal{N}(\mu, \sigma^2) \\ z_{t+1}|j &\sim \mathcal{N}(j\theta, j\delta^2) \\ j &\geq 0 \text{ has probability } e^{-\omega}\omega^j/j! \end{aligned}$$

- Parameter values
  - Match mean and variance of log consumption growth
  - Jump probability ($\omega = 0.01$), mean ($\theta = -0.3$), and variance ($\delta^2 = 0.015^2$) [similar to Barro, Nakamura, Steinsson, and Ursua]

Macro disasters: entropy

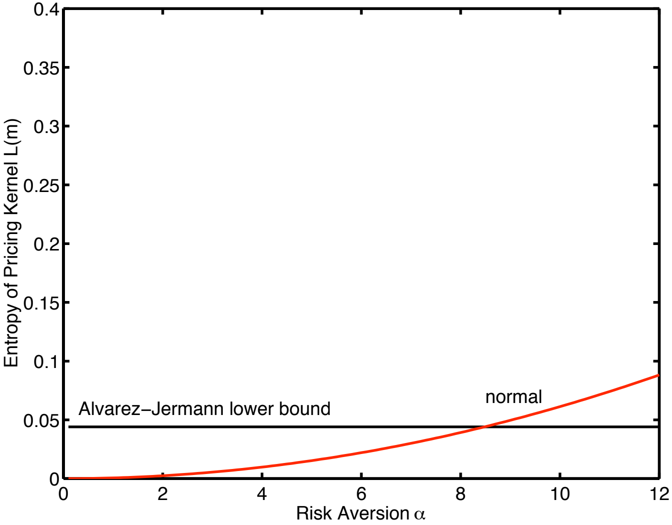
- Cumulant generating functions

$$\begin{aligned} k(s;\log g) &= s\mu + s^2\sigma^2/2 + \omega\left(e^{s\theta+s^2\delta^2/2} - 1\right) \\ k(s;\log m) &= s\log\beta - s\alpha\mu + s^2\alpha^2\sigma^2/2 + \omega\left(e^{-s\alpha\theta+s^2\alpha^2\delta^2/2} - 1\right) \end{aligned}$$

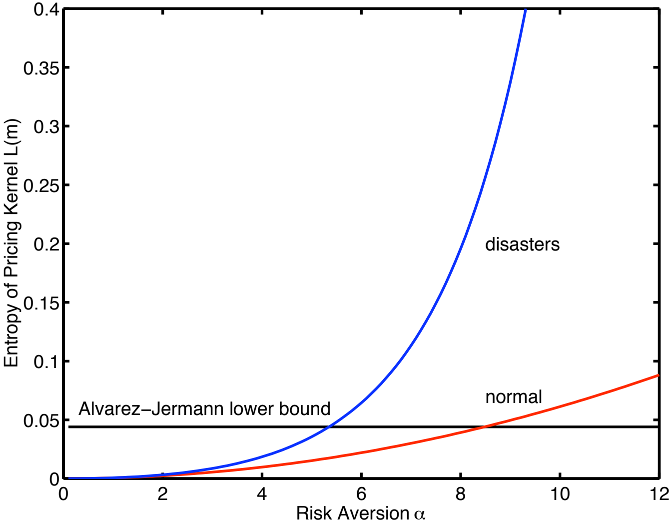
- Entropy

$$L(m) = \alpha^2\sigma^2/2 + \omega\left(e^{-\alpha\theta+\alpha^2\delta^2/2} - 1\right) + \alpha\omega\theta,$$

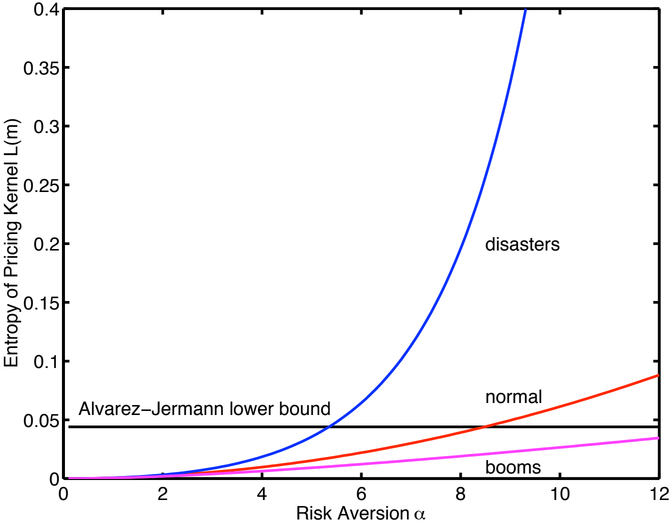
Macro disasters: entropy



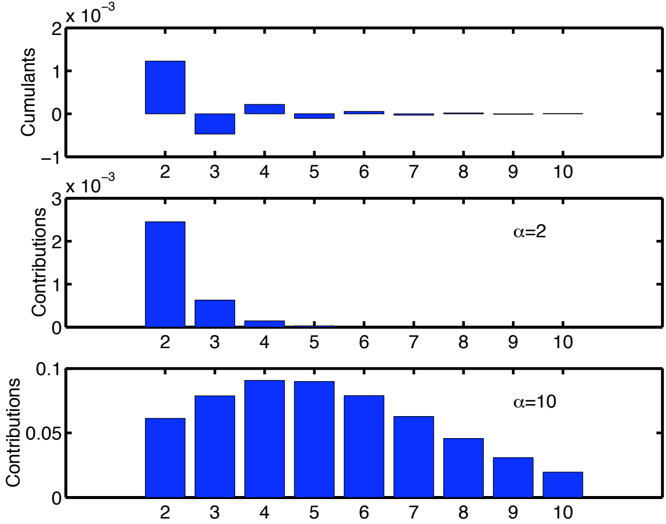
Macro disasters: entropy



Macro disasters: entropy



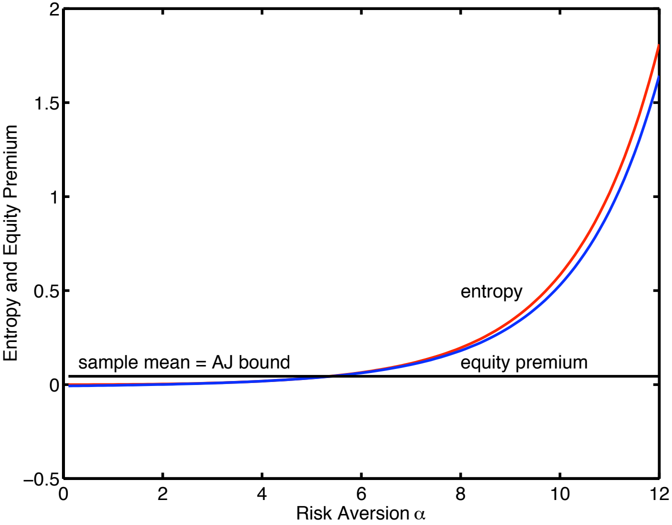
Macro disasters: cumulants



Macro disasters: cumulants

			High-Order Cumulants	
Model ($\alpha = 10$)	**Entropy**	Variance/2	Odd	Even
Normal	**0.0613**	0.0613	0	0
Poisson disaster	**0.5837**	0.0613	0.2786	0.2439
Poisson boom	**0.0266**	0.0613	**–0.2786**	0.2439

Macro disasters: equity premium



Option disasters: overview

- Options an obvious source of information, but ...
  - Options on equity, not consumption
  - Determine risk-neutral, not true distribution
  - True distribution has the usual lack of data problems
- Plan of attack
  - Estimate risk-neutral distribution from options
  - Estimate true distribution two ways
  - Compare options implied by macro-based disaster model

Risk-neutral probabilities: examples

- Normal log consumption growth
  - If $\log g \sim \mathcal{N}(\mu, \sigma^2)$ (true distribution)
  - Then risk-neutral distribution also lognormal with $\mu^* = \mu - \alpha\sigma^2, \sigma^* = \sigma$
- Poisson log consumption growth
  - Jumps have probability ω and distribution $\mathcal{N}(\theta, \delta^2)$
  - Risk-neutral distribution has same form with $\omega^* = \omega \exp[-\alpha\theta + (\alpha\delta)^2/2], \theta^* = \theta - \alpha\delta^2, \delta^* = \delta$

Option disasters: information in option prices

- Put option (bet on low returns)

$$q_t^p = \frac{1}{R_f}\mathbb{E}_t^*(b - r_{t+1}^e)^+$$

- Strategy
  - Estimate p^* by varying strike price b (cross section)
  - Estimate p and R_f from time series data
- Black-Scholes-Merton benchmark
  - Quote prices as implied volatilities (high price $\Leftrightarrow$ high vol)
  - Horizontal line if lognormal
  - "Skew" suggests disasters

Option disasters: Merton model

- Equity returns i.i.d.

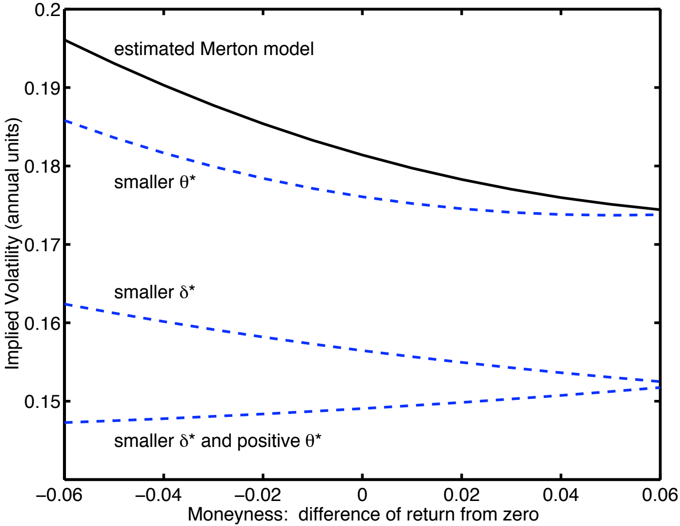
$$\begin{aligned}
\log r^e_{t+1} &= \log r^1 + w_{t+1} + z_{t+1} \\
w_{t+1} &\sim \mathcal{N}(\mu, \sigma^2) \\
z_{t+1}|j &\sim \mathcal{N}(j\theta, j\delta^2) \\
j &\geq 0 \text{ has probability } e^{-\omega}\omega^j/j!
\end{aligned}$$

- Risk-neutral distribution: ditto with *s

Option disasters: parameter values

- Choose $(\mu, \sigma, \omega, \theta, \delta)$ to match distribution of equity returns
  - Jumps: $\omega = 1.512$, $\theta = -0.0259$, $\delta = 0.0229$
  - Equity premium: $\mu + \omega\theta$
  - Variance of equity returns: $\sigma^2 + \omega(\theta^2 + \delta^2)$
- Set $(\omega^*, \theta^*, \delta^*)$ to match option prices
  - Jumps: $\omega^* = \omega$, $\theta^* = -0.0482$, $\delta^* = 0.0981$
  - Set $\sigma^* = \sigma$
  - Set μ^* to satisfy pricing relation $(1/r_f)\mathbb{E}^* r^e = 1$
- All of this from Broadie, Chernov, and Johannes (JF, 2007)

Option disasters: implied volatility for 3mo options



Option disasters: components of entropy

			High-Order Cumulants	
Model	Entropy	Variance/2	Odd	Even
Consumption-based models				
Normal ($\alpha = 10$)	0.0613	0.0613	0	0
Poisson ($\alpha = 10$)	0.5837	0.0613	0.2786	0.2439
Poisson ($\alpha = 5.38$)	0.0449	0.0177	0.0173	0.0099
Option-based model				
Option model	**0.7647**	0.4699	**0.1130**	**0.1819**

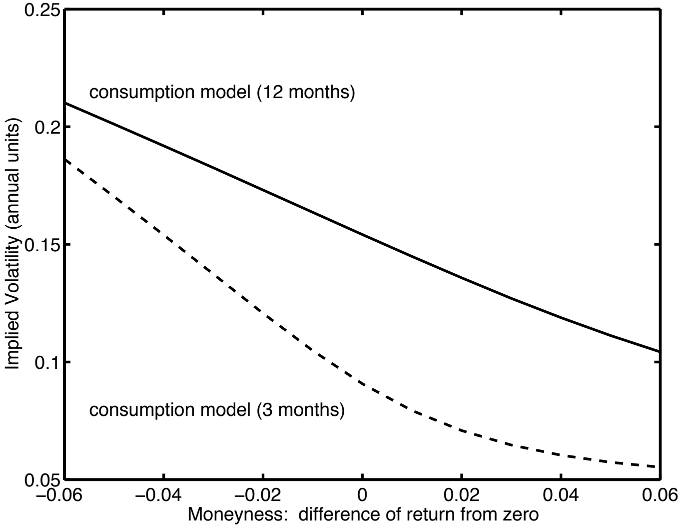
Comparing macro- and option-based models

- Direct comparison of entropy and cumulants
- Consumption growth implied by option prices
  - Scale option-based p^* to consumption
  - Find p using power utility
  - Result: more modest skewness and kurtosis, tail probabilities
- Option prices implied by consumption growth
  - Find macro-based p^* using power utility
  - Scale to equity returns
  - Compute option prices
  - Result: steeper volatility smile

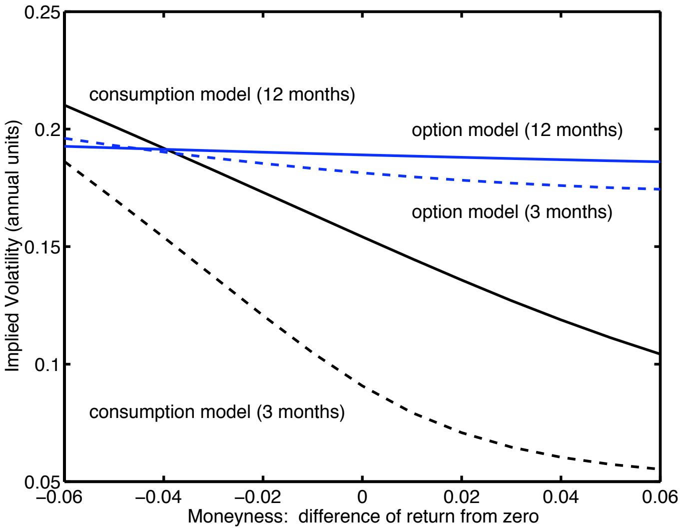
Comparing models: consumption implied by options

	Consumption Process Based on	
	Cons Growth	Option Prices
α	5.38	10.07
ω	0.0100	1.3864
θ	–0.3000	–0.0060
δ	0.1500	0.0229
Skewness	–11.02	–0.31
Excess Kurtosis	145.06	0.87
Tail prob (≤ -3 st dev)	**0.0090**	**0.0086**
Tail prob (≤ -5 st dev)	**0.0079**	**0.0002**

Comparing models: options implied by consumption



Comparing models: options implied by consumption



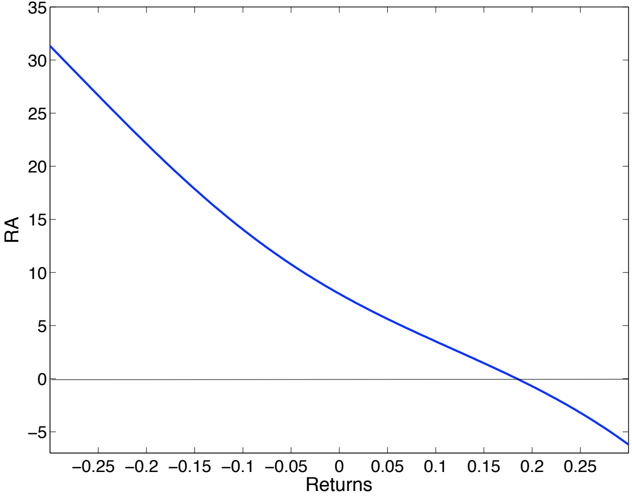
Risk aversion in the option model

- "Risk aversion" implied by arbitrary pricing kernel

$$\text{RA} \;\equiv\; -\frac{\partial \log m}{\partial \log g} \;=\; -\frac{\partial \log(p^*/p)}{\partial \log r^e} \cdot \frac{\partial \log r^e}{\partial \log g}$$

- Risk aversion not constant ("state dependent")
- Parameters imply greater aversion to adverse risks

Risk aversion in the option model



Bottom line

- Barro et al; Longstaff & Piazzesi; Rietz
  - Back out asset pricing implications (p^*) from assumptions on preferences (m) and real-world probability distributions (p)
  - Disasters contribute to equity premium, entropy
  - Evident in macro data
- We look at options
  - Estimate p from time series of market returns and p^* from cross-section of option prices
  - Implies very high entropy
  - Smile/smirk suggests something like disasters
  - But more modest than macro data

Open questions

- Sources of apparent risk aversion
  - Exotic preferences
  - Heterogeneous agents
  - Examples: Alvarez, Atkeson, and Kehoe; Bates; Chan and Kogan; Du; Guvenen; Lustig and Van Nieuwerburgh; Longstaff and Wang
- Consumption and dividends
  - Examples: Bansal and Yaron, Gabaix, Longstaff and Piazzesi
- Time dependence
  - Short rate, predictable returns, stochastic volatility
  - Examples: Drechsler and Yaron, Wachter, Shaliastovich