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Online Appendix to "Why are Buyouts Levered? The Financial Structure of Private Equity Funds"*

B. Proofs of Propositions 1, 3, 5, and 6 for the General Case.

Proof of Proposition 1: In this proof, we are careful about showing that buying and holding publicly traded securities should be disallowed. That this is optimal also for other forms of capital raising is easy to show, but we simply assume it in the rest of the proofs.

In each period and state, the GP decides whether or not to seek financing. Financing entails a contract $\{w, T\}$ where w is a security satisfying monotonicity and limited liability, and $T \in \{A, N\}$ specifies whether trading in public market assets is allowed (A) or not (N). We assume public market assets to be zero NPV, and to have a full support of cash flows: Any random variable $x_i \geq 0$ satisfying $E(x_i) = I$ can be purchased for I in the public markets.

If the GP seeks financing, the investor then chooses whether to accept and supply financing I, or deny financing in which case the game ends. If the investor accepts, the GP

*Citation format: Axelson, Ulf, Per Strömberg, and Michael S. Weisbach, Internet Appendix for "Why Are Buyouts Levered? The Financial Structure of Private Equity Funds," Journal of Finance DOI: 10.1111/j.1540-6261.2009.01473.x. Please note: Wiley-Blackwell is not responsible for the content or functionality of any supporting information supplied by the authors. Any queries (other than missing material) should be directed to the authors of the article.

then decides whether to invest in a firm, the risk-free asset, or some public market asset i (if $T = A$).

There can never be a separating equilibrium where different types of GPs seek financing with different contracts $\{w, T\}$. Since the investor never breaks even on a security issued by a fly-by-nighter or a GP with a bad project, those types will always have an incentive to mimic a good type.

In period 1, the static equilibrium with $T = N$, $w_I(I) = I$, and $w_I(Z)$ such that:

$$((1 - \alpha) p + \alpha) w_I(Z) \geq I.$$

so that investors break even, is the unique financing equilibrium since it is the only one that does not leave any rent to fly-by-night operators. However, in period 2, investors will know that any GP who invested in a real firm is not a fly-by-night operator. In period 2, it is therefore possible that contracts may be such that $w_I(I) < I$ or trading in public assets is allowed. But this would be inconsistent with the assumption that fly-by-night operators do not invest in period 1 because they would have an incentive in period 1 to mimic real GPs by investing in a wasteful project, so that they can earn positive rents in period 2. Thus, in any period, the on-equilibrium path cannot involve contracts in which fly-by-night operators earn a positive rent. This shows that if any financing equilibrium exists in any period, it is the same as the static solution. It remains to show that the repeated static solution in fact exists as a dynamic equilibrium.

Suppose the static solution is played in period 1. In the low state, there is no financing, which means that in period 2 fly-by-night operators are not screened out, so the static solution is again an equilibrium. In the high state, there is financing, so fly-by-night operators are screened out. We now state the Intuitive Criterion that then has to be satisfied for the static solution to be a financing equilibrium in period 2. (The general definition can be found in Cho and Kreps (1987); We state the particular version that applies to our setting). The static solution is a financing equilibrium satisfying the intuitive criterion if and only if there is *no* contract $\{w', T'\}$ where the security design w' satisfies monotonicity and limited liability, such that:

1. Investors would be willing to finance the deal in exchange for w' if they believe the issuing GP is good:

$$w'_I(Z) \geq I.$$

2. GPs finding bad firms are *strictly worse off* issuing w' than they are in the postulated equilibrium, even if investors are willing to finance the deal in exchange for w' : If $T' = N$,

$$\max\left(I - w'_I(I), p\left(Z - w'_I(Z)\right)\right) < p\left(Z - w_I(Z)\right).$$

If $T' = A$,

$$\max_i E(x_i - w'_I(x_i)) < p\left(Z - w_I(Z)\right).$$

3. GPs finding good firms are *strictly better off* issuing w' than they are in the postulated equilibrium if investors are willing to finance the deal in exchange for w':

$$w'_I(Z) < w_I(Z).$$

If there were such a contract $\{w', T'\}$, and it was issued out of equilibrium, we assume that investors would conclude that the issuing GP must be good. If investors have that belief, good GPs would indeed be better off issuing contract $\{w', T'\}$, so $\{w, N\}$ cannot be an equilibrium. (To rule out $\{w, N\}$ as an equilibrium, it is essential that there is a $\{w', T'\}$ that is only preferred by GPs finding good firms. If we cannot rule out that GPs finding bad firms might also be better off if financed by $\{w', T'\}$, investors could rationally believe that anyone offering $\{w', T'\}$ out of equilibrium is bad, so that a best response could be to not supply financing for $\{w', T'\}$.)

We show that there is no contract such that Conditions 3 and 2 are satisfied at the same time. For Condition 3 to be satisfied, we need $w'_I(Z) < w_I(Z)$. But then,

$$\max\left(I - w'_I(I), p\left(Z - w'_I(Z)\right)\right) \geq p\left(Z - w'_I(Z)\right) > p\left(Z - w_I(Z)\right).$$

This rules out contracts where $T' = N$. On the other hand, if $T' = A$, there is always a traded asset x_i such that $x_i = 0$ with probability $1 - p'$ and $x_i = Z$ with probability p', where $p' = \frac{I}{Z} > p$. Therefore, we have that

$$\max_i E(x_i - w'_I(x_i)) \geq p'\left(Z - w'_I(Z)\right) > p\left(Z - w_I(Z)\right).$$

This rules out contracts where $T' = A$. Hence, the static solution is an equilibrium in period 2 if it was played in period 1. ■

Proof of Proposition 3: Here, we show the following stronger result than Proposition 3 as stated in the printed version:

EXTENDED PROPOSITION 3: *In the pure ex ante financing case, when $Z \leq 2I$, the GP captures all the surplus if $p \leq \frac{1}{2}$ or if $p > \frac{1}{2}$ and:*

$$\left(\frac{E(\alpha)}{1-E(\alpha)}\right)^2 \leq \frac{(1-p)I}{(Z-I)\left(2-\frac{1}{p}\right)}.$$

When $Z > 2I$, the GP captures all the surplus if:

$$\left(\frac{E(\alpha)}{1-E(\alpha)}\right)^2 \geq \frac{(1-p)I}{Z-I}.$$

Otherwise, the LP can get a strictly positive surplus.

Proof: Case 1: $Z \leq 2I$. For this case, $w_{GP}(Z) = 0$ from the fly-by-night condition, so that the IC constraint becomes:

$$w_{GP}(Z+I) \geq p w_{GP}(2Z).$$

First, suppose $p \leq \frac{1}{2}$. Note that if we set:

$$\begin{aligned} w_{GP}(Z+I) &= k(Z-I), \\ w_{GP}(2Z) &= k2(Z-I), \end{aligned}$$

for $k \in [0,1]$ the IC constraint is satisfied since $p < \frac{1}{2}$. Then, there is always a k such that LPs just break even if the social surplus is positive, since at $k = 1$ they do not break even and at $k = 0$ they get the whole social surplus. Thus, the GP captures all the surplus.

Now, suppose $p > \frac{1}{2}$. Suppose we set:

$$\begin{aligned} w_{GP}(Z+I) &= Z - I, \\ w_{GP}(2Z) &= \frac{Z-I}{p}. \end{aligned}$$

Note that this contract satisfies monotonicity when $p > \frac{1}{2}$. The contract also maximizes GP rent among contracts that satisfy incentive compatibility. Therefore, if the break-even constraint of the LP is slack at this contract, LPs will earn strictly positive rents for any incentive compatible contract. At this contract, the break-even constraint is slack if:

$$E(\alpha)^2 \left(2Z - \frac{Z-I}{p}\right) + \left(2E(\alpha)(1-E(\alpha)) + (1-E(\alpha))^2 p\right) 2I + (1-E(\alpha))^2 (1-p) I > 2I,$$

which can be rewritten as:

$$E(\alpha)^2 \left(2Z - \frac{Z-I}{p}\right) + (1-E(\alpha))^2 (1-p) I > \left(E(\alpha)^2 + (1-E(\alpha))^2 (1-p)\right) 2I.$$

Dividing by $(1-E(\alpha))^2$ and gathering terms gives the condition as:

$$\left(\frac{E(\alpha)}{1-E(\alpha)}\right)^2 > \frac{(1-p) I}{(Z-I)\left(2-\frac{1}{p}\right)}.$$

If this condition is not satisfied, it is easy to see that there is an x such that a contract with $w_{GP}(Z+I) = x \leq Z - I$ and $w_{GP}(2Z) = \frac{x}{p}$ makes the LP just break even, so in that case the GP captures all the surplus. This proves the first part of the proposition.

Case 2: $Z > 2I$. For this case, we have to have $p < \frac{1}{2}$ for Condition 1 to be satisfied.

Suppose we set $w_{GP}(Z+I) = Z - I$ and, according to Claim 1 in the proof of Proposition 2, $w_{GP}(Z) = Z - 2I$. The break-even constraint of the LP then becomes

$$E(\alpha)^2 (2Z - w_{GP}(2Z)) + \left(2E(\alpha)(1-E(\alpha)) + (1-E(\alpha))^2 p\right) 2I + (1-E(\alpha))^2 (1-p) I \geq 2I.$$

Suppose we force this to hold with equality and solve for $w_{GP}(2Z)$:

$$E(\alpha)^2 (2Z - w_{GP}(2Z)) + (1-E(\alpha))^2 (1-p) I = \left(E(\alpha)^2 + (1-E(\alpha))^2 (1-p)\right) 2I$$

$$\Leftrightarrow$$

$$w_{GP}(2Z) = 2(Z-I) - \left(\frac{1-E(\alpha)}{E(\alpha)}\right)^2 (1-p) I. \quad \text{(B1)}$$

For monotonicity not to be violated, $w_{GP}(2Z)$ as defined above must be higher than $Z - I$:

$$(Z-I) \leq 2(Z-I) - \left(\frac{1-E(\alpha)}{E(\alpha)}\right)^2 (1-p) I. \quad \text{(B2)}$$

Rewriting, this corresponds to the second inequality in the proposition. Suppose this condition holds. We now show that the IC constraint is satisfied for this contract. Plugging in for $w_{GP}(2Z)$ from above, the IC constraint is satisfied if:

$$\begin{aligned} Z - I \geq & \frac{1 + \frac{1-E(\alpha)}{E(\alpha)} 2p}{1 + \frac{1-E(\alpha)}{E(\alpha)} p} (1-p)(Z-2I) \\ & + p\left(2(Z-I) - \left(\frac{1-E(\alpha)}{E(\alpha)}\right)^2 (1-p) I\right). \end{aligned}$$

Taking the derivative of the right-hand side with respect to $x \equiv \frac{1-E(\alpha)}{E(\alpha)}$ gives:

$$\frac{2p\,(1+xp) - p\,(1+2xp)}{(1+xp)^2}\,(1-p)\,(Z-2I) - 2px\,(1-p)\,I, \tag{B3}$$

which has the same sign as:

$$\frac{Z-2I}{(1+xp)^2} - 2xI.$$

This is decreasing in x. Thus, if it is negative for the lowest possible x, it is always negative.

The lowest possible $x \equiv \frac{1-E(\alpha)}{E(\alpha)}$ is derived from Condition 1 as:

$$x = \frac{Z-I}{I-Zp}.$$

Plugging this into Expression (B3) gives:

$$\begin{aligned} & \frac{Z-2I}{(1+xp)^2} - 2\frac{Z-I}{I-Zp}I \\ = \; & \frac{Z-2I}{(1+xp)^2} - 2\frac{Z-I}{1-\frac{Z}{I}p} < 0. \end{aligned}$$

Thus, the derivative w.r.t. to $\frac{1-E(\alpha)}{E(\alpha)}$ is everywhere negative, and we should set $\frac{1-E(\alpha)}{E(\alpha)}$ as low as possible to make it hard to satisfy the IC constraint.

Plugging $\frac{1-E(\alpha)}{E(\alpha)} = \frac{Z-I}{I-Zp}$ into the IC constraint gives:

$$\begin{aligned} Z-I \;\; \geq \;\; & \frac{1+\frac{Z-I}{I-Zp}2p}{1+\frac{Z-I}{I-Zp}p}\,(1-p)\,(Z-2I) \\ & +p\left(2\,(Z-I) - \left(\frac{Z-I}{I-Zp}\right)^2 (1-p)\,I\right). \end{aligned}$$

Dividing by p, this can be rewritten as:

$$\left(\frac{Z-I}{I-Zp}\right)^2 (1-p)\,I + \frac{I-Zp}{p} > (Z-2I)\left(\frac{Z-I}{I}\right).$$

Noting that:

$$\frac{Z-I}{I-Zp} > \frac{Z-I}{I},$$

it is harder to satisfy the inequality if we divide the LHS by $\frac{Z-I}{I-Zp}$ and the RHS with $\frac{Z-I}{I}$, which gives:

$$\begin{aligned}\left(\frac{Z-I}{I-Zp}\right)(1-p)\,I + \frac{\frac{I-Zp}{p}}{\frac{Z-I}{I-Zp}} &> Z-2I \Longleftrightarrow \\ (Z-I)\,\frac{1-p}{1-\frac{Z}{I}p} + \frac{\frac{I-Zp}{p}}{\frac{Z-I}{I-Zp}} &> Z-2I.\end{aligned}$$

This always holds, since

$$\frac{1-p}{1-\frac{Z}{I}p} > 1.$$

Thus, the IC constraint is always satisfied when the investor just breaks even, which shows that the GP captures the whole surplus.

Finally, when Condition B2 does not hold, it is possible to show that a feasible contract sometimes must leave strictly positive rents to LPs (proof available upon request). ■

Proof of Proposition 5 for the general case where Assumption 1 does not necessarily hold: First, when we abandon Assumption 1, the IC condition in Lemma 1 is no longer sufficient. We first show the IC condition that is necessary and sufficient for the general case:

LEMMA 2: *A necessary and sufficient condition for a contract* $w_{GP}(x)$ *to be incentive compatible in the mixed ex ante and ex post case is*

$$\begin{aligned} & q\left(\alpha_H+(1-\alpha_H)p\right)w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)p}+K\right) \\ > \; & E(\alpha)\left(pw_{GP}\left(2\left(Z-(I-K)\right)\right)+(1-p)\,w_{GP}\left(Z-(I-K)\right)\right)+(1-E(\alpha))* \\ & p\max\left[w_{GP}\left(Z-(I-K)+K\right),pw_{GP}\left(2\left(Z-(I-K)\right)\right)+2\left(1-p\right)w_{GP}\left(Z-(I-K)\right)\right] \end{aligned} \tag{B4}$$

Proof: Using the definitions of x_{GB}, x_{GG}, and x_{GG} in (A7), if the GP invested in a good firm in period 1, he will pass up a bad firm if:

$$w_{GP}(x_{GB}) > pw_{GP}(x_{GG})+(1-p)\,w_{GP}\left(\frac{1}{2}x_{GG}\right). \tag{B5}$$

The last term is the case where the bad firm does not pay off, and the fund defaults on its period 2 ex post debt. We also have to check the off-equilibrium behavior where the GP invested in a bad firm in period 1. If the GP invested in a bad firm in period 1 he will pass up a bad firm in period 2 if:

$$pw_{GP}(x_{GB})+(1-p)\,w_{GP}(K) > p^2w_{GP}(x_{GG})+p(1-p)\,w_{GP}\left(\frac{1}{2}x_{GG}\right)+(1-p)\,pw_{GP}\left(\frac{1}{2}x_{GG}\right).$$

The two last terms are, respectively, the case where the first bad firm pays off and the second does not, and the case where the first bad firm does not pay off and the second does.

Since $w_{GP}(K) = 0$ from the fly-by-night condition, this can be rewritten as:

$$w_{GP}(x_{GB}) > pw_{GP}(x_{GG}) + 2(1-p)\,w_{GP}\left(\frac{1}{2}x_{GG}\right). \tag{B6}$$

Note that this is a stricter condition than Condition (B5). Now consider the GP's investment incentives in period 1. In period 1, it is always optimal to invest in a good project. We must check that the GP does not want to invest in a bad project to sustain the separating equilibrium. The condition for this is:

$$\begin{aligned} q(\alpha_H + (1-\alpha_H)p)\,w_{GP}(x_{BG}) &> E(\alpha)\left(pw_{GP}(x_{GG}) + (1-p)\,w_{GP}\left(\frac{1}{2}x_{GG}\right)\right) \\ &+ (1-E(\alpha))\,p\max\left(w_{GP}(x_{GB}), pw_{GP}(x_{GG}) + 2(1-p)\,w_{GP}\left(\frac{1}{2}x_{GG}\right)\right). \end{aligned}$$

The last line is the GP payoff when he has invested in a bad firm in period 1 and encounters another bad firm in period 2, in which case he will either invest in it or not, depending on whether Condition (B6) holds or not. Note that this condition implies Condition (B5), since $w_{GP}(x_{BG}) \leq w_{GP}(x_{GB})$ and:

$$\begin{aligned} &\frac{E(\alpha)}{q(\alpha_H + (1-\alpha_H)p)}\left(pw_{GP}(x_{GG}) + (1-p)\,w_{GP}\left(\frac{1}{2}x_{GG}\right)\right) \\ &+\frac{(1-E(\alpha))\,p}{q(\alpha_H + (1-\alpha_H)p)}\max\left(w_{GP}(x_{GB}), pw_{GP}(x_{GG}) + 2(1-p)\,w_{GP}\left(\frac{1}{2}x_{GG}\right)\right) \end{aligned}$$

$$\begin{aligned} &\geq \frac{(E(\alpha) + (1-E(\alpha))\,p)}{q(\alpha_H + (1-\alpha_H)p)}\left(pw_{GP}(x_{GG}) + (1-p)\,w_{GP}\left(\frac{1}{2}x_{GG}\right)\right) \\ &\geq pw_{GP}(x_{GG}) + (1-p)\,w_{GP}\left(\frac{1}{2}x_{GG}\right). \end{aligned}$$

Thus, the only relevant incentive constraint is the period 1 IC constraint. ■

Proof that K^* is set maximal at $I - (\alpha_L + (1-\alpha_L)\,p)\,Z$: First, we have to have $x_{BG} > \frac{1}{2}x_{GG}$ and $x_{BG} > 2K$ for the equilibrium to be feasible, or else the IC condition will not be satisfied. Suppose this is true, so that cash-flow states are ordered as

$$x_{GG} > x_{GB} > x_{BG} > \max\left(\frac{1}{2}x_{GG}, 2K\right) > K.$$

Suppose contrary to the claim in the proposition that $K < K^*$ at some candidate optimal contract w_I satisfying monotonicity and limited liability. Now suppose we increase K by Δ arbitrarily small, increase $w_I(K)$ by Δ, increase $w_I(2K)$ by 2Δ, increase $w_I\left(\frac{1}{2}x_{GG}\right)$ by

$$\begin{array}{ll} \Delta \text{ if} & w_I\left(\frac{1}{2}x_{GG}\right) = \frac{1}{2}x_{GG}, \\ 2\Delta \text{ if} & w_I\left(\frac{1}{2}x_{GG}\right) < \frac{1}{2}x_{GG}, \end{array}$$

and increase $w_I(x_{BG}), w_I(x_{GB})$, and $w_I(x_{GG})$ by $B \in \left(2\Delta, \Delta + \frac{\Delta}{\alpha_H + (1-\alpha_H)p}\right)$ such that the break-even constraint and the maximand are unchanged:

$$\begin{aligned} & (B - 2\Delta)\left(E(\alpha)^2 + E(\alpha)(1 - E(\alpha)) + (1 - E(\alpha))\,q\,(\alpha_H + (1-\alpha_H)\,p)\right) \\ = \; & \Delta\,(1 - E(\alpha))\,q\,(1-\alpha_H)(1-p)\,. \end{aligned}$$

Note that for small Δ, these changes do not violate monotonicity or the fly-by-night condition. However, the IC constraint is weakly relaxed, since $w_{GP}(x_{BG})$ goes up weakly and $w_{GP}(x_{GB})$ and $w_{GP}(x_{GG})$ go down weakly. Hence, the problem is relaxed, and we can increase K without loss of generality. Thus, there is no loss of generality from setting $K = K^*$

in an optimal contract. ■

We now state a stronger result about optimal contracts than Proposition 5 and prove it for the general case.

EXTENDED PROPOSITION 5: *Suppose* $Z-(I-K^*) \leq 2K^*$. *The optimal investor security* $w_I(x)$ *is debt with face value* $F = w_I\left(Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right) \in \left[2K^*, Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right)$ *plus a carry* $k(\max(x-S,0))$ *starting at* $S \in \left[Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*, Z-(I-K^*)+K^*\right]$. *For* $F = 2K^*$, *we have* $k \in (0,1), S \in \left[Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*, Z-(I-K^*)+K^*\right]$ *and for* $F > 2K^*$, *we have* $k = 1$ *(call option) and* $S = Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*$. *For a fixed expected value* $E(w_I(x))$ *given to investors,* F *is set minimal.*

Suppose $Z-(I-K^*) > 2K^*$. *The optimal investor security* $w_I(x)$ *is debt with face value* $F = w_I\left(Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right) \in \left[2K^*, Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right)$ *plus a carry* $k(\max(x-S,0))$ *starting at* $S \in \left[Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*, Z-(I-K^*)+K^*\right]$. *For* $S < Z-(I-K^*)+K^*$, *we have* $k = 1$ *(call option), and for* $S = Z-(I-K^*)+K^*$, *we have* $k \in (0,1)$.

Proof: **Case 1:** $Z-(I-K^*) \leq 2K^*$. This is the case when the GP gets no pay-off if he fails with one project, so $w_{GP}(Z-(I-K^*)) = 0$. For this case, the IC condition (B4) reduces to:

$$
\begin{aligned}
& q(\alpha_H + (1-\alpha_H)p)\, w_{GP}\left(Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right) \\
> \; & E(\alpha) p w_{GP}(2(Z-(I-K^*))) \\
& + (1-E(\alpha))\, p \max\left(w_{GP}(Z-(I-K^*)+K^*), p w_{GP}(2(Z-(I-K^*)))\right).
\end{aligned}
$$

Given a certain expected pay-off $E(x - w_{GP}(x))$ to investors, the optimal contract should relax the IC condition maximally without violating the fly-by-night condition or the monotonicity constraints. Any decrease of $w_{GP}(2(Z - (I - K^*)))$ or $w_{GP}(Z - (I - K^*) + K^*)$ and increase of $w_{GP}\left(Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right)$ that keeps the expected value of the security constant relaxes the constraint. First, suppose $w_I\left(Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right) > 2K^*$. The optimal contract in the proposition then claims that:

$$\begin{aligned} w_I\ (Z - (I - K^*) + K^*) &= w_I\left(Z - \frac{I - K^*}{\alpha_H + (1-\alpha_H)p} + K^*\right) + (I - K^*) - \frac{I - K^*}{\alpha_H + (1-\alpha_H)p}, \\ w_I\ (2(Z - (I - K^*))) &= w_I\ (Z - (I - K^*) + K^*) + Z - I. \end{aligned}$$

Suppose this is not true. First, suppose:

$$\begin{aligned} w_I\ (Z - (I - K^*) + K^*) &< w_I\left(Z - \frac{I - K^*}{\alpha_H + (1-\alpha_H)p} + K^*\right) + (I - K^*) - \frac{I - K^*}{\alpha_H + (1-\alpha_H)p}, \\ w_I\ (2(Z - (I - K^*))) &\leq w_I\ (Z - (I - K^*) + K^*) + Z - I. \end{aligned}$$

Then, we can increase $w_I\ (Z - (I - K^*) + K^*)$ and decrease $w_I\left(Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right)$ (which means we decrease $w_{GP}(Z - (I - K^*) + K^*)$ and increase $w_{GP}\left(Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right)$) to keep the break-even constraint and the maximand constant without violating monotonicity. This relaxes the IC constraint and so improves the contract.

Now, suppose:

$$\begin{aligned} w_I\ (Z - (I - K^*) + K^*) &= w_I\left(Z - \frac{I - K^*}{\alpha_H + (1-\alpha_H)p} + K^*\right) + (I - K^*) - \frac{I - K^*}{\alpha_H + (1-\alpha_H)p}, \\ w_I\ (2(Z - (I - K^*))) &< w_I\ (Z - (I - K^*) + K^*) + Z - I. \end{aligned}$$

Then, we can increase $w_I\left(2\left(Z-\left(I-K^*\right)\right)\right)$ by ε and decrease $w_I\left(Z-\left(I-K^*\right)+K^*\right)$ and $w_I\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$ by:

$$\frac{\varepsilon E\left(\alpha\right)^2}{E\left(\alpha\right)\left(1-E\left(\alpha\right)\right)+\left(1-E\left(\alpha\right)\right)q\left(\alpha_H+\left(1-\alpha_H\right)p\right)}$$

to keep the break-even constraint and the maximand constant without violating monotonicity. This relaxes the IC constraint and so improves the contract.

Next suppose $w_I\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)=2K^*$. Then, $w_I\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$ cannot be lowered without violating the fly by night condition.

First, note that increasing $w_I\left(2\left(Z-\left(I-K^*\right)\right)\right)$ by ε and reducing $w_I\left(Z-\left(I-K^*\right)+K^*\right)$ by:

$$\varepsilon\frac{E\left(\alpha\right)}{\left(1-E\left(\alpha\right)\right)}$$

to keep the break-even constraint constant leaves the IC constraint unchanged if:

$$w_{GP}\left(Z-\left(I-K^*\right)+K^*\right)>pw_{GP}\left(2\left(Z-\left(I-K^*\right)\right)\right),$$

and relaxes it if:

$$w_{GP}\left(Z-\left(I-K^*\right)+K^*\right)<pw_{GP}\left(2\left(Z-\left(I-K^*\right)\right)\right).$$

Therefore, if such a transfer does not violate monotonicity, it (weakly) relaxes the IC constraint. Thus, a contract that maximally relaxes the IC constraint keeping the expected value $E(w)$ constant should have:

$$w_I\ (2(Z-(I-K^*)))=w_I\ (Z-(I-K^*)+K^*)+Z-I$$

if $w(X(Z,K^*))>2K^*$. However, for such a contract we have:

$$\begin{aligned} pw_{GP}(2(Z-(I-K^*))) &= p[2(Z-(I-K^*))-(w_I\ (Z-(I-K^*)+K^*))+Z-I] \\ &= pw_{GP}(Z-(I-K^*)+K^*) \\ &< w_{GP}(Z-(I-K^*)+K^*), \end{aligned}$$

and therefore the IC constraint is unchanged if we lower $w_I\ (2(Z-(I-K^*)))$ and increase $w_I\ (Z-(I-K^*)+K^*)$ slightly so that

$$w_I\ (2(Z-(I-K^*)))=w_I\ (Z-(I-K^*)+K^*)+k(Z-I),$$

where $k<1$. Thus, this contract can be expressed as a carry. This proves the first part of the Proposition.

Case 2: $Z-(I-K^*)>2K^*$. This is the case when the GP can get some pay-off even if he fails with one project, so it is possible to have $w_{GP}(Z-(I-K^*))>0$. It is always optimal to set $w_I(Z-(I-K^*))$ as high as possible at $\min\left(Z-(I-K^*),w_I\ \left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)\right)$, so the contract will have a debt piece as before with face value $w_I\ \left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$. However, it is no longer true that we want to set this face value as low as possible given a fixed $E(w_I\ )$ by increasing the higher pay offs. This is because when we reduce the face value, we also increase the pay off to the GP if he fails with one and succeeds with one firm, which can worsen incentives. To establish the Proposition, we start with the following Lemma:

LEMMA 3: $w_I\ (Z-(I-K^*)) = \min\left(Z-(I-K^*), w_I\ \left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)\right).$

Proof. First, note that given $w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$, the highest we can set $w_A(Z-(I-K^*))$ is the expression in the lemma from monotonicity and the fact that $Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^* > Z-(I-K^*)$ in feasible contracts. Suppose $w_A(Z-(I-K^*))$ is lower than this upper bound. Then, we can increase it without changing the break even constraint and the maximand, since the outcome $Z-(I-K^*)$ does not happen in equilibrium. This relaxes the IC constraint and so improves the contract. ■

This proves that the first piece is debt with face value $w_I\ \left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right).$

Next, suppose $w_I\ (Z-(I-K^*)+K^*) > w_I\ \left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$. Then, the proposition states that:

$$w\left(2\left(Z-(I-K^*)\right)\right) = w_I\ \left(Z-(I-K^*)+K^*\right)+Z-I$$

which is the highest possible value for $w_I\ (2(Z-(I-K^*)))$ given $w_I\ (Z-(I-K^*)+K^*).$ Suppose this is not the case. Then, we can lower $w_I\ (Z-(I-K^*)+K^*)$ and increase w_I $(2(Z-(I-K^*)))$ to keep the break-even constraint and the maximand constant without violating monotonicity. If:

$$w_{GP}\left(Z-(I-K^*)+K^*\right) > p w_{GP}\left(2\left(Z-(I-K^*)\right)\right)+2p(1-p)\, w_{GP}\left(Z-(I-K^*)\right),$$

this does not change the IC constraint, but if:

$$w_{GP}\left(Z-(I-K^{*})+K^{*}\right) < pw_{GP}\left(2\left(Z-(I-K^{*})\right)\right)+2p\left(1-p\right)w_{GP}\left(Z-(I-K^{*})\right),$$

the IC constraint is relaxed and so this improves the contract. ■

Proof of Proposition 6. We state the general version here. It is easy to verify that this reduces to the version in the paper when Assumption 1 holds:

EXTENDED PROPOSITION 6: *Necessary and sufficient conditions for the equilibrium to be implementable are that it creates social surplus, that:*

$$q\left(\alpha_H+\left(1-\alpha_H\right)p\right)\geq p,$$

and that:

$$\frac{\alpha_L+\left(1-\alpha_L\right)p}{\alpha_H+\left(1-\alpha_H\right)p} < \min\left(\frac{I}{Z}, 1-\frac{I}{Z}+\alpha_L+\left(1-\alpha_L\right)p\right).$$

Proof: First, it is necessary that $x_{BG} > 2K$, or else the lefthand side of the IC condition (B4) is zero from monotonicity. Second, it is necessary that $x_{BG} > \frac{1}{2}x_{GG}$, since otherwise $w_{GP}\left(x_{BG}\right) \leq w_{GP}\left(\frac{1}{2}x_{GG}\right)$. This would violate the IC condition (B4), since in that case the righthand side of the IC condition becomes:

$$\begin{aligned}&E(\alpha)\left[pw_{GP}\left(x_{GG}\right)+\left(1-p\right)w_{GP}\left(\frac{1}{2}x_{GG}\right)\right]\\&+\left(1-E(\alpha)\right)p\max\left(w_{GP}\left(x_{GB}\right), pw_{GP}\left(x_{GG}\right)+2\left(1-p\right)w_{GP}\left(\frac{1}{2}x_{GG}\right)\right)\end{aligned}$$

$$\geq (E(\alpha) + (1 - E(\alpha))\,p) \left[p w_{GP}\left(x_{GG}\right) + (1-p)\, w_{GP}\left(\frac{1}{2} x_{GG}\right)\right] \geq (E(\alpha) + (1 - E(\alpha))\,p)\, w_{GP}\left(x_{BG}\right).$$

Since $E(\alpha) + (1 - E(\alpha))\,p > q\left(\alpha_H + (1 - \alpha_H)\,p\right)$, this is larger than the lefthand side of the IC condition.

The two necessary conditions above can be rewritten as:

$$\frac{I - K}{\alpha_H + (1 - \alpha_H)\,p} < Z - K,$$

and:

$$\frac{I - K}{\alpha_H + (1 - \alpha_H)\,p} < I.$$

Note that both these are easier to satisfy for higher K, and by setting K maximal at K^* from Proposition 5, the conditions become:

$$\frac{\alpha_L + (1 - \alpha_L)\,p}{\alpha_H + (1 - \alpha_H)\,p} Z < Z - \left(I - \left(\alpha_L + (1 - \alpha_L)\,p\right) Z\right),$$

and:

$$\frac{\alpha_L + (1 - \alpha_L)\,p}{\alpha_H + (1 - \alpha_H)\,p} Z < I.$$

These conditions together give the last expression in the proposition.

The first part of the proposition is proved as follows. The righthand side of Condition (B4) is given by:

$$E(\alpha)\left(p w_{GP}\left(x_{GG}\right) + (1-p)\, w_{GP}\left(\frac{1}{2} x_{GG}\right)\right)$$
$$+ (1 - E(\alpha))\, p \max\left(w_{GP}\left(x_{GB}\right), p w_{GP}\left(x_{GG}\right) + 2\,(1-p)\, w_{GP}\left(\frac{1}{2} x_{GG}\right)\right)$$

$$\geq E(\alpha)pw_{GP}\left(x_{GG}\right)+\left(1-E(\alpha)\right)pw_{GP}\left(x_{GB}\right)\geq pw_{GP}\left(x_{BG}\right),$$

where the last step follows from monotonicity. Therefore, the IC condition can only be satisfied if:

$$q\left(\alpha_H+\left(1-\alpha_H\right)p\right)\geq p.$$

Thus, this is a necessary condition for the equilibrium to be implementable. To show that it together with the other conditions are sufficient, suppose they are satisfied. Then, for ε small enough, it is always possible to set:

$$w_{GP}\left(\frac{1}{2}x_{GG}\right)=0,\quad w_{GP}\left(x_{GG}\right)=w_{GP}\left(x_{BG}\right)=w_{GP}\left(x_{GB}\right)=\varepsilon.$$

For this contract, the IC condition reduces to:

$$q\left(\alpha_H+\left(1-\alpha_H\right)p\right)\geq p.$$

For ε small enough, investors always break even as long as social surplus is created. ■

C. Extra Results Not in the Printed Version.

PROPOSITION 7: *Suppose pure ex post financing is feasible in the high state:*

$$\left(\alpha_H+\left(1-\alpha_H\right)p\right)Z\geq I.$$

Then, even when the most efficient mixed financing equilibrium can not be implemented, the following mixed financing equilibrium can always be implemented:

1. GPs invest in both good and bad firms in period 1, but ex ante capital K per period is set so that financing is possible only in the high state.

2. In the second period, GPs who did not invest in period 1 only get financing in the high state, and invest in both good and bad firms. GPs who did invest in period 1 get financing in both the high and the low state, and invest efficiently.

Proof: Set the ex ante capital K per period as:

$$K < I - (\alpha_L + (1 - \alpha_L)\, p)\, Z.$$

Given the postulated equilibrium investment behavior, this assures that GPs who have not yet invested cannot raise the required ex post capital $I - K$ in the low state. In the high state, the required face value of ex post debt will be:

$$F = \frac{I - K}{\alpha_H + (1 - \alpha_H)\, p}.$$

For GPs who have invested in the first period, suppose the market assumes that investments made in the second period are good. Then, the required face value of debt will be $I - K$. We need to make sure that a GP who has invested in the first period indeed has an incentive to invest efficiently in the second period. It is easy to show that GPs who find good firms in the second period will always invest. The condition for a GP who invested in a good firm in period 1 not to invest in a bad firm in period 2 is given by:

$$\begin{aligned} w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}+K\right) &\geq pw_{GP}\left(2Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}-(I-K)\right) \\ &\quad +(1-p)w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}\right). \end{aligned}$$

The condition for a GP who invested in a bad firm in period 1 not to invest in a bad firm in period 2 is given by:

$$\begin{aligned} w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}+K\right) &\geq pw_{GP}\left(2Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}-(I-K)\right) \\ &\quad +(1-p)w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}\right) \\ &\quad +(1-p)w_{GP}\left(Z-(I-K)\right). \end{aligned}$$

Note that this is a stronger condition and therefore necessary and sufficient for incentive compatibility. Note that if we can set $w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)p}+K\right)=a>0$, we can always make this condition hold by setting:

$$w_{GP}\left(2Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}-(I-K)\right)=a,$$

$$w_{GP}\left(Z-(I-K)\right)\leq a,$$

and:

$$w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}\right)=0.$$

Furthermore, if the equilibrium generates social surplus, there is always an $a>0$ such that investors break even. From the fly-by-night constraint, we can only set $a > 0$ if:

$$Z - \frac{I - K}{\alpha_H + (1 - \alpha_H) p} + K > 2K,$$

i.e., if:

$$K (1 - \alpha_H) (1 - p) > I - Z (\alpha_H + (1 - \alpha_H) p).$$

But since we have assumed that $I - Z (\alpha_H + (1 - \alpha_H) p) < 0$, this holds automatically for $K > 0$. Hence, we can structure the contract such that GPs who invested in period 1 invest efficiently in period 2. ■

PROPOSITION 8: *As the number of periods T goes to infinity, a pure ex ante financing contract with $w_{GP}(x) = 0$ for $x \leq TI$ and $w_{GP}(x) = k_T (x - TI)$ for $x > TI$ implements investment behavior arbitrarily close to the first best, and the GP captures arbitrarily close to the full surplus.*

Proof: Denote by $\gamma_t \in \{G, B\}$ the type of firm that arrives in period t, and by $a_t \in \{i, n\}$ the decision by the GP to invest (i) or not (n) in the firm. Also, denote by $\Gamma_t = \{\sigma_t\}_j^t$ and $A_t = \{a_t\}_j^t$ the history of firm arrivals and investment decisions up to period t. The investment strategy in period t is then given as a function $a_t (A_{t-1}, \Gamma_{t-1}, \gamma_t)$. It is obvious that it is optimal for the GP to invest in all good firms, so $a_t (A_{t-1}, \Gamma_{t-1}, H) = i$ for all Γ_{t-1}, A_{t-1}. Denote the per-period unconditional probability that the GP invests in a bad firm by δ_T:

$$\delta_T = \frac{1}{T} \sum_{\Omega_T} \Pr(\Omega_T) \sum_{t=1}^{T} 1_{\sigma_t = B} * 1_{a_t(A_{t-1}, \Gamma_{t-1}, B) = i}.$$

We prove the proposition by showing that $\lim_{T\to\infty} \delta_T = 0$ for the optimal GP strategy. First, denote the pay-off to the fund net of invested capital from good investments by Π_T^G, given by:

$$\Pi_T^G = \sum_{t=1}^{T} E(\alpha)(Z - I).$$

As the arrival rate of good firms is independent, the pay-off per period from good firm investments converges in probability to $E(\alpha)(Z - I)$: For any $\varepsilon > 0$, we have that:

$$\lim_{T\to\infty} \Pr\left(\alpha(Z - I) + \varepsilon \geq \frac{\Pi_T^G}{T} \geq \alpha(Z - I) - \varepsilon\right) = 1.$$

Now suppose $\lim_{T\to\infty} \delta_T = \delta > 0$. Denote the set of arrival histories for which the number of bad investments is of order $O(T)$ by B:

$$B = \left\{\lim_{T\to\infty} \Gamma_T : \lim_{T\to\infty} \frac{\sum_{t=1}^{T} 1_{\gamma_t = B} * 1_{a_t(A_{t-1}, \Gamma_{t-1}, B) = i}}{T} > 0\right\}.$$

Then, we must have $\Pr(B) > 0$. Furthermore, for each path, denote by $\Pi_T^B(\Gamma_T)$ the pay-off net of invested capital to the fund from bad investments:

$$\Pi_T^B(\Gamma_T) = \sum_{t=1}^{T} 1_{\gamma_t = B} * 1_{a_t(A_{t-1}, \Gamma_{t-1}, B) = i} * (pZ - I).$$

For paths in B, as the outcome of the bad investments are independent draws, the pay-off per period converges: For any $\varepsilon > 0$, we have that for $\lim_{T\to\infty} \Gamma_T \in B$,

$$\lim_{T\to\infty} \Pr\left(\frac{\Pi_T^B(\Gamma_T)}{T} < 0\right) = 1.$$

For paths in the complement of B, the number of bad firm investments are of order $o(T)$, so the per period pay-off goes to zero. But then, we have the per-period total payoff as:

$$\Pi_T = \lim_{T \to \infty} \left(\frac{\Pi_T^G}{T} + \frac{\Pi_T^B}{T} \right),$$

and:

$$\lim_{T \to \infty} \Pr \left(\frac{\Pi_T}{T} < \alpha \left(Z - I \right) \right) = 1.$$

But this means that the GP pay-off per period is strictly lower than $\lim_{T \to \infty} k_T \alpha \left(Z - I \right)$, which is what he gets if $\delta = 0$. Hence, we must have $\delta = 0$, so the GP follows the efficient investment policy with probability one. Therefore, the LP breaks even as long as $k < 1$. The optimal contract for the GP in which the LP breaks even then must have:

$$\lim_{T \to \infty} k_T = 1.$$

Therefore, the GP captures arbitrarily close to all the surplus in the limit. ■