Internet Appendix for "Incomplete-Market Equilibria Solved Recursively on an Event Tree"*

I. Theorem 1

THEOREM 1: *Given a price process S and initial wealths $W_{l,0}$, the choice of consumption plans c_l, trading strategies θ_l and state prices ϕ_l maximizes investor l's goal at all times and in all possible states of the economy if and only if the three equations (6), except for the third one when $t = T$, are satisfied for any $0 \leq t \leq T$ and in any state $\xi \in \mathcal{F}_t$. Furthermore, the value functions $V_{l,t}$, treated as functions of the entering wealths $W_{l,t}$ for time t, are concave in any state $\xi \in \mathcal{F}_t$, for any $0 \leq t \leq T$, and, if it exists, the solution (c_l, ϕ_l) is necessarily unique.*

Proof. Given that the principle of dynamic programming applies, if the consumption plan c_l and the trading strategy θ_l attain agent-l's objective, then one can claim that $x^* = c_{l,t}$ and $y^* = \theta_{l,t}$ solve the "primal" optimization problem:

$$\begin{aligned} &\max_{x,y}\ G_{l,t}(x,y) \triangleq U_{l,t}(x) + \mathbb{E}_t\left[V_{l,t+1}\left[y\cdot(S_{t+1}+\delta_{t+1})\right]\right] \\ &\quad\text{subject to: } x + y\cdot S_t = e_{l,t} + W_{l,t}, \quad x\in\mathbb{R}_{++},\ y\in\mathbb{R}^N. \end{aligned} \tag{IA.1}$$

Since $x > 0$, that is, consumption is strictly positive, the Lagrangian for this problem is given by

$$\begin{aligned} \mathcal{L}_{l,t}(x,y,\lambda) = G_{l,t}(x,y) + \lambda\times\left(e_{l,t} + W_{l,t} - x - y\cdot S_t\right), \\ x\in\mathbb{R}_{++},\ y\in\mathbb{R}^N,\ \lambda\in\mathbb{R}\,. \end{aligned}$$

Since the left side of the only constraint in (IA.1) is a linear function of $(x,y)\in\mathbb{R}_{++}\times\mathbb{R}^N$ with gradient (treated as a vector column) $\nabla(x + y\cdot S_t) = \{1, S_t\} \in \mathbb{R}^{1+N}$, first-order conditions (4) imply the following relation

$$\nabla G_{l,t}(c_{l,t},\theta_{l,t}) = \phi_{l,t}\times\{1,S_t\}. \tag{IA.2}$$

Consider next the quantities $c_{l,t}$, $\theta_{l,t}$ and $\phi_{l,t}$, which are defined implicitly from (4) (we assume that $\theta_{l,T} = \mathbf{0}$), as functions of the entering wealth $W_{l,t}$. After differentiating both sides in (IA.2) we get

$$\nabla^2 G_{l,t}\left[c_{l,t}(W_{l,t}),\theta_{l,t}(W_{l,t})\right]\cdot\{c'_{l,t}(W_{l,t}),\theta'_{l,t}(W_{l,t})\} = \phi'_{l,t}(W_{l,t})\times\{1,S_t\},$$

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and this implies that

$$\left\{c'_{l,t}(W_{l,t}), \theta'_{l,t}(W_{l,t})\right\}^{\intercal} \cdot \nabla^2 G_{l,t}\left[c_{l,t}(W_{l,t}), \theta_{l,t}(W_{l,t})\right] \cdot \left\{c'_{l,t}(W_{l,t}), \theta'_{l,t}(W_{l,t})\right\}$$
$$= \phi'_{l,t}(W_{l,t}) \times \left(c'_{l,t}(W_{l,t}) + S_t \cdot \theta'_{l,t}(W_{l,t})\right) = \phi'_{l,t}(W_{l,t}), \tag{IA.3}$$

where we have used the identity

$$c'_{l,t}(W_{l,t}) + \theta'_{l,t}(W_{l,t}) \cdot S_t = 1\,, \tag{IA.4}$$

which is obtained by differentiating both sides of the constraint

$$c_{l,t}(W_{l,t}) + \theta_{l,t}(W_{l,t}) \cdot S_t = e_{l,t} + W_{l,t}.$$

For some fixed $1 \leq l \leq L$, consider the entire system of first-order conditions (6) at all nodes $\xi \in \mathcal{F}_t$, $0 \leq t \leq T$. It is clear from the terminal condition $V_{l,T}\left(W_{l,T}\right) = U_{l,T}\left(c_{l,T}\right) \equiv U_{l,T}\left(e_{l,T} + W_{l,T}\right)$ that the value function $V_{l,T}(\cdot)$ is strictly concave in any state $\sigma \in \Omega$, that is, all value functions $V_{l,T,\sigma}(\cdot)$, $\sigma \in \Omega$, are strictly concave. Now suppose that for some $0 \leq t < T$ one can claim that the value functions $V_{l,t+1,\eta}(\cdot)$ are strictly concave, for all possible choices of $\eta \in \xi^+$ and $\xi \in \mathcal{F}_t$. Then the function

$$\mathbb{R}_{++} \times \mathbb{R}^N \ni (x, y) \longrightarrow G_{l,t}(x, y) \in \mathbb{R},$$

which was defined in (IA.1), also must be strictly concave in state $\xi \in \mathbb{F}_t$. Since the security prices are nonnegative, (IA.4) implies that the vector

$$\left\{c'_{l,t}(W_t), \theta'_{l,t}(W_t)\right\} \in \mathbb{R}^{1+N}\,,$$

cannot vanish. In conjunction with the strict concavity of $G_{l,t}(\cdot,\cdot)$, (IA.3) and (5) imply that in state $\xi \in \mathbb{F}_t$ one must have

$$V''_{l,t}\left(W_t\right) = \phi'_{l,t}\left(W_t\right) < 0.$$

The fact that the value functions $V_{l,t,\xi}(\cdot)$, $\xi \in \mathcal{F}_t$, are strictly concave for any $0 \leq t \leq T$ now follows by induction. As a result, we can claim that all functions $G_{l,t,\xi}(\cdot,\cdot)$ are strictly concave and therefore the first-order conditions in (4) are both necessary and sufficient and cannot be satisfied with more than one choice for (c_l, θ_l, ϕ_l), $1 \leq l \leq L$. Finally, taking into account (5), these first-order conditions can be written in the form (6).

II. Multiplicity of Solutions in the Heaton and Lucas (1996) model

When solving the Heaton and Lucas (1996) example of Section VI.A, using the actual data of their Table 2, page 455, we encounter multiplicity when we go beyond seven points in time ($T = 6$). Adding one more period ($T = 7$) leads to the results (for period $t = 0$), illustrated in Figures IA.1 and IA.2. These figures clearly show multiple solutions of the system of Euler equations associated with two particular exogenous states (state 1 and state 5) at time $T - 7$ for a small range of values of the ω state variable.

Figure IA.1 displays group 2's share of future consumption in the eight successor states in relation to group 2's share of consumption when the time-0 state is state 1. Observe first that the relationships are not monotonic: when group 2's share today in state 1 is reduced to about 0.025, its share of consumption in future state 8 suddenly rises all the way to 0.8 while its amount of consumption in future state 4 suddenly drops more moderately. Referring back to the Markov-chain data in the article by Heaton and Lucas (1996), Table 2, page 455, one notices that the probability of a transition from state 1 to state 8 is very low and equal to 0.002.[1]

Group 2 can afford a huge share of consumption in state 8 in exchange for a moderate drop in state 4 only because state 8 has extremely low probability of occurring. The observation illustrates a basic principle of incomplete financial markets: very large transfers of consumption from one group to another occur endogenously in low-probability events.

Figure IA.2 displays the same graphs as Figure IA.1 but on a much larger scale for the interval $[0, 0.05]$ of group 2's consumption share today. It is apparent that the relationships for most of the future states display a jump combined with a "beak" or a "cusp." Over a small interval of today's consumption values, for a given value of that consumption, there exist three possible solutions for tomorrow's consumption in the various states, which means that future consumption is not a function of today's consumption; it is generally a correspondence.

The equilibrium "hesitates" between letting group 2 consume in state 8 or in state 4.[2] The source of the phenomenon is not the same as in the Basak-Cuoco example in which the two agents have different risk aversions. The source is the nonmonotonicity of the eight stock price functions (not shown), which are U-shaped in each of the eight future states. The multiplicity in the Heaton-Lucas case could not occur at time $T - 1$. As one lengthens

[1]The Heaton-and-Lucas Markov chain is symmetric: swapping groups 1 and 2 and swapping simultaneously states 1-4 with states 5-8 gives back the same process. While we focus our observations on the transitions from state 1, and especially that from state 1 to state 8, there are symmetric observations to be made about transitions from state 5 and especially that from state 5 to state 4.

[2]Actually, the multiplicity of the solutions of the Euler equations, described above, was discovered somewhat by accident. After seven steps in the backward induction applied to the Heaton-Lucas example, the root-finding procedure failed to report a solution to the system of Euler equations associated with state 1, for certain values of the state variable ω close to one, and in state 5, for values of ω close to one. Neither increasing the working precision nor using a denser grid seemed to matter. It was the lengthy investigation of this failure that led to the discovery of multiple solutions for a small range of values of the state variables near the endpoints of the domain ($[0, 1]$). Apparently, with our choice of starting point for the search and step-size, the root-finding procedure (based on the Newton-Raphson method) was getting lost between three different solutions.

the horizon, the stock being a security that lives to the horizon behaves more and more differently from the short-term interest rate. The stock price U-shaped functions become flatter and take a sharper turn towards low and high shares of consumption.

If $T - 7$ were the initial period, state 1 were the initial exogenous state of the economy, and the initial endowment with securities were such that the initial value of ω fell in that small range, then the global system of Euler equations would have three different sets of solutions, that is, the economy would have three distinct global equlibria.

If $T-7$ is not the initial period, we simply do not have tools at our disposal to determine whether there exists an initial endowment – say, at time $T - 100$ – with which the induction could be closed at that time (that is, the initial endowment would be feasible) and that would force the agents 92 periods later to exit state 1 or state 5 with a portfolio that puts the state variable ω in the problematic interval. If no such initial endowment existed for period $T - 100$, then the multiplicity in states 1 and 5 in period $T - 7$ would be irrelevant. If, however, there were an initial endowment for period $T - 100$ that leads to that interval in period $T-7$, then the global system would admit multiple solutions and this multiplicity would have to be dealt with one way or another.

III. The 2D Manifold of Judd, Kubler and Schmedders (2002)

To illustrate the discussion on dimensions, we obtain, for the first set of parameters of Section VI.A above, the 2D manifold of solutions of the system (19) and display it in Figure IA.3. Please observe that the graph is a ruled surface, made of horizontal straight segments, so the 2D manifold is really of one dimension.[3] Furthermore, this ruled surface is of very small and variable width; it is defined on a narrow strip that is the set of portfolios for which current consumption remains strictly positive. That strip, as mentioned, is not known until after the solution is computed because it depends on securities prices.

IV. REFERENCES

Heaton, John and Deborah Lucas, 1996, Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing, *Journal of Political Economy* 104, 443-487.

Judd, Kenneth L., Felix Kubler and Karl Schmedders, 2002, A Solution Method for Incomplete Asset Markets with Heterogeneous Agents, working paper, Universität Zürich.

[3]That property of the manifold explains why, when running simulations of the resulting equilibrium, Judd, Kubler and Schmedders (2002) find that the path of the economy is confined to a 1D curve, such as their Figure 2, page 24.

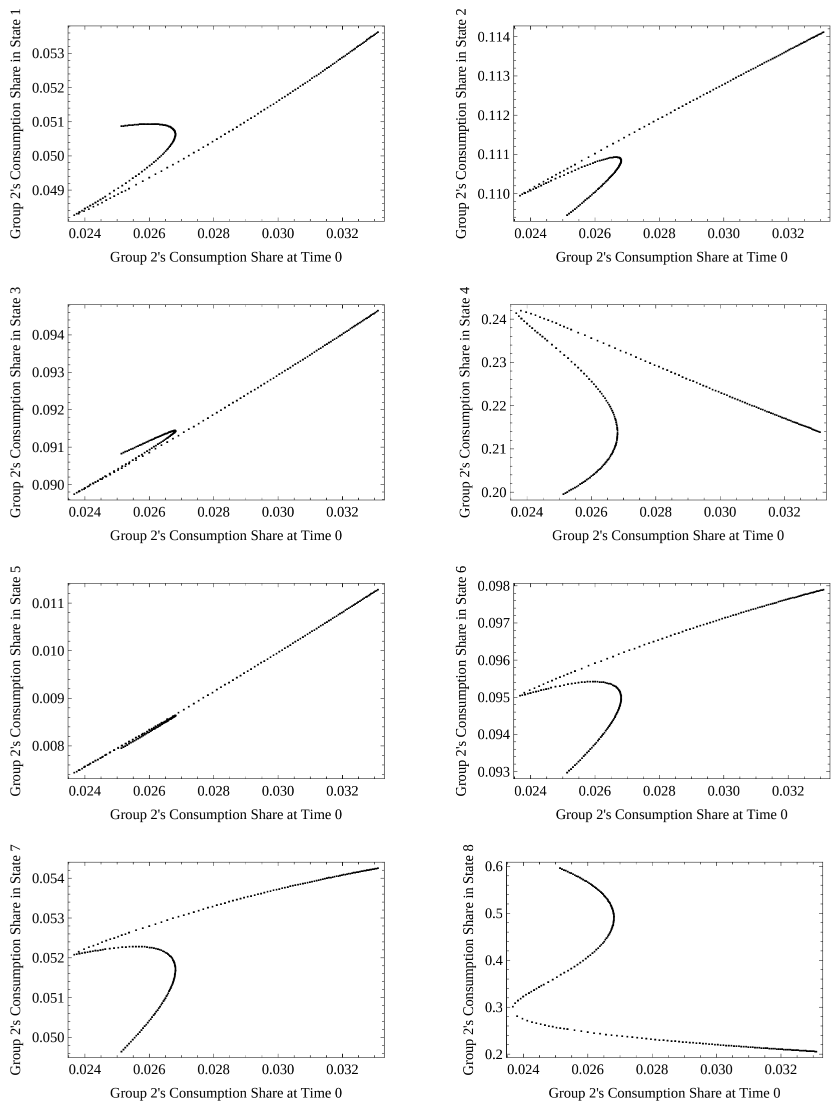

Figure IA.1. Heaton-and-Lucas example, consumption correspondence. This figure shows on the y-axis the consumption share of group 2 in each of the eight states at time 1 when the time-0 state is state 1. On the x-axis is the fraction of output consumed by group 2 at time 0. Parameter values are as in Heaton and Lucas (1996), Table 2, page 455 and $T = 7$, $\gamma = -0.5$. Note: here the x-grid has been chosen to focus the diagram on the singular point towards the left of the x-axis. The grid normally used for computations is more evenly spaced.

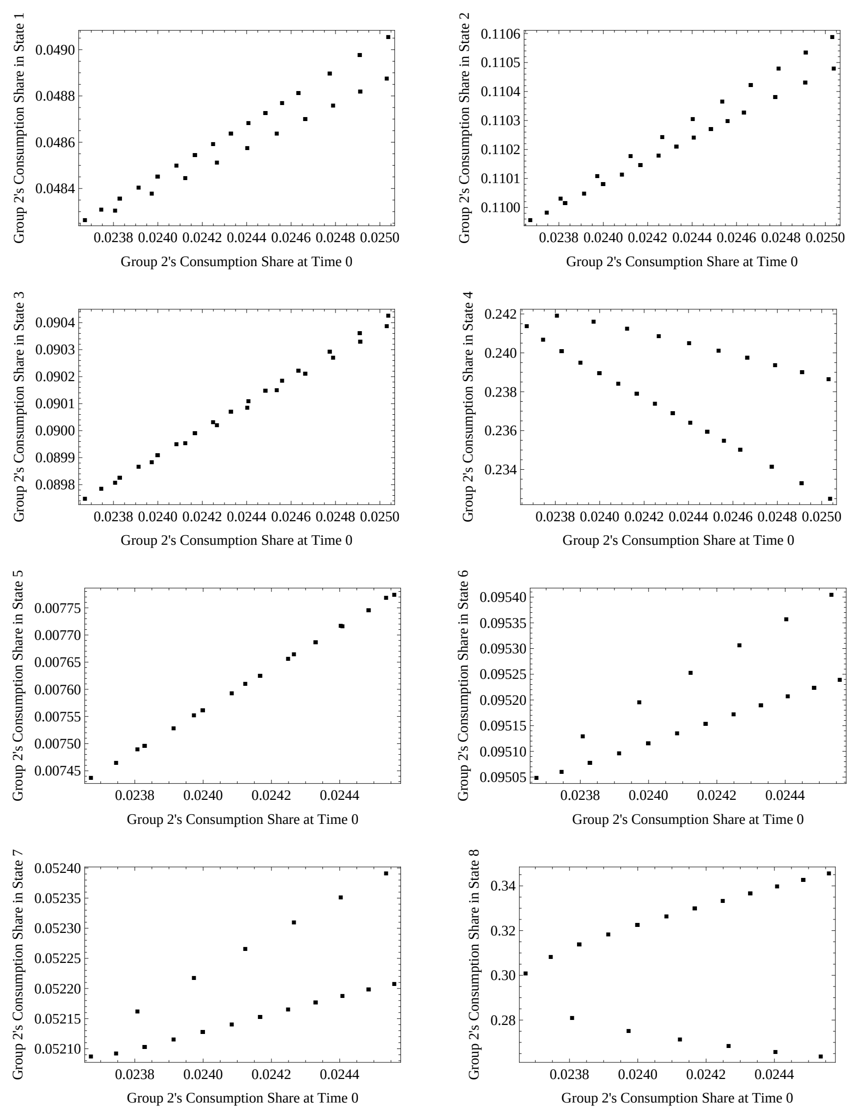

Figure IA.2. Heaton-and-Lucas example, consumption correspondence (large scale). This figure shows on the y-axis the consumption share of group 2 in each of the eight states at time 1 when the time-0 state is state 1. On the x-axis is the fraction of output consumed by group 2 at time 0. Parameter values are as in Figure IA.1.

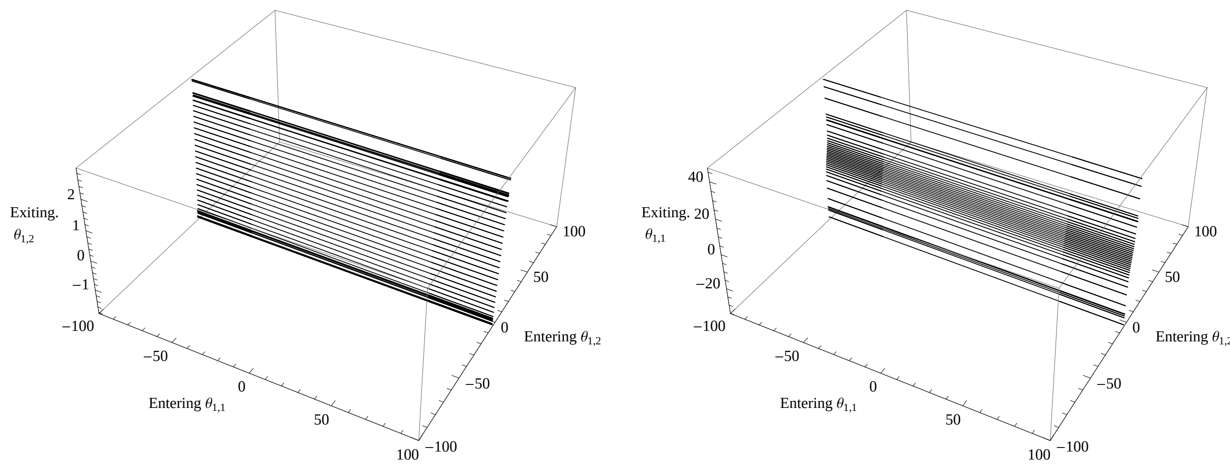

Figure IA.3. Eight-state Markov-chain example. This figure shows the portfolio choices of investors of group 1 when the two groups of agents only trade the risk-free asset and equity in the first state of the economy. On the x, y-axes are the entering (i.e., pre-trade portfolios). On the z-axis are the exiting (i.e., post-trade) portfolios. $\theta_{1,1}$ is group 1's share of the riskless security; $\theta_{1,2}$ is the same group's share of the risky security. Parameter values are as in Figure 9.