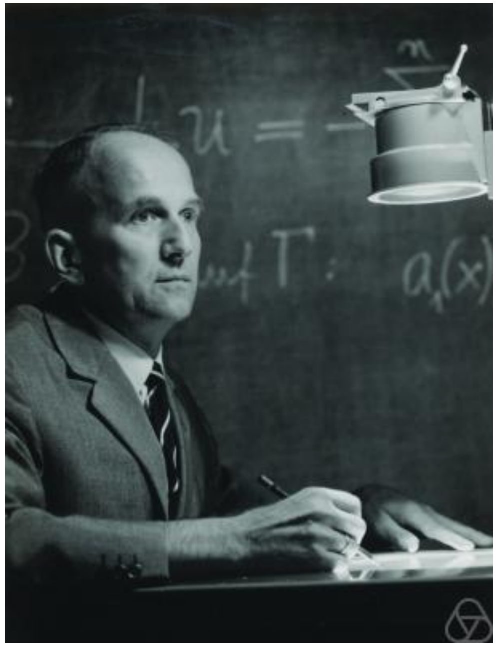

What is the Collatz Conjecture and Why Is It so Interesting?

Alexander Atwood

Suffolk County Community College

AMATYC Annual Conference

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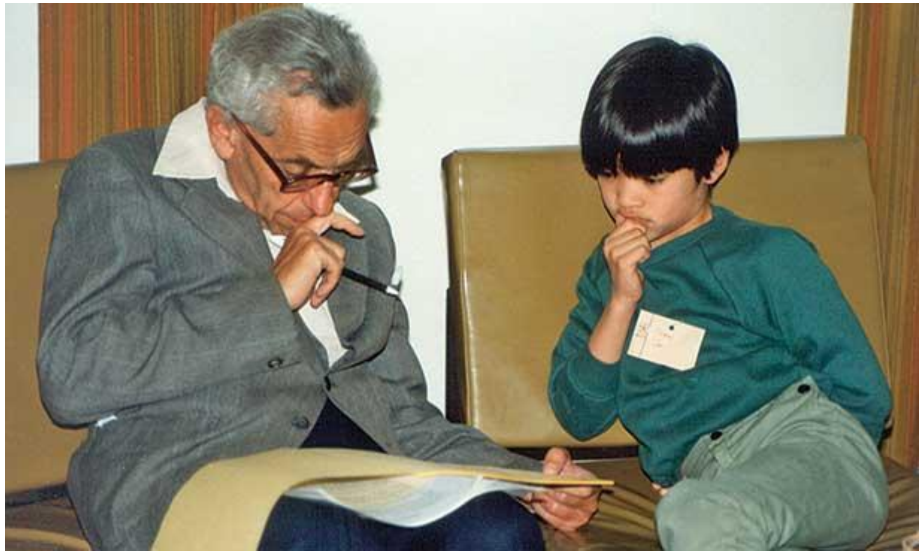
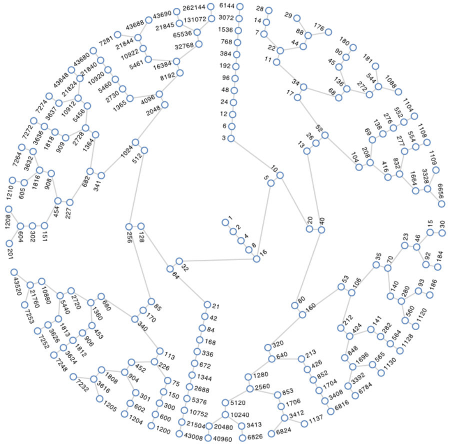


Proposed by Lothar Collatz in the 1930's, the Collatz Conjecture is one of the most difficult open problems in mathematics. We will describe the conjecture, demonstrate how it works, talk about why proving it is so difficult, and describe recent significant work by Terence Tao on this subject.



Overview of Presentation

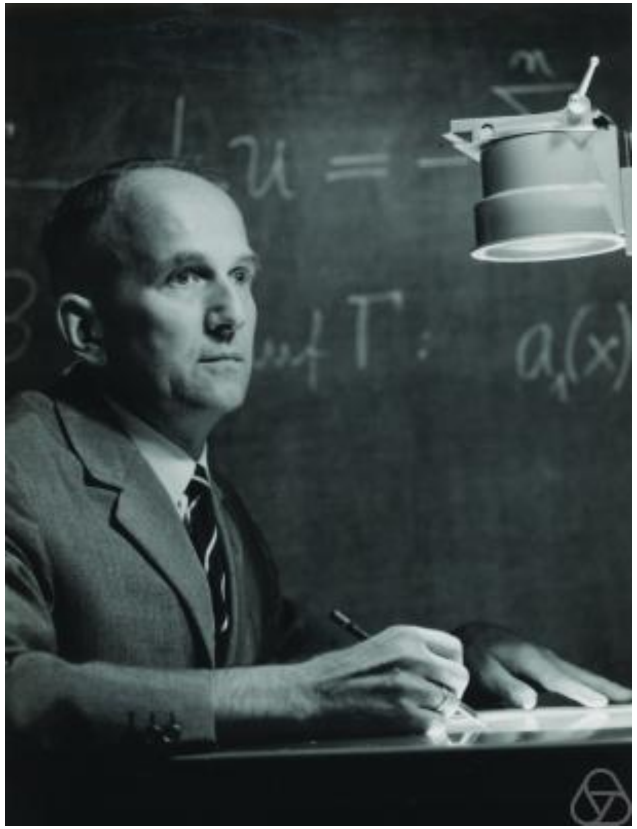
- Mathematical Statement of the Collatz Conjecture
- Illustration of the Collatz Conjecture
- Brief history of the Conjecture
- How one might try to prove the Conjecture
- Why the Conjecture is so difficult to prove
- The important work of Terence Tao (2019 paper)
- Why the Conjecture is interesting for our students
- References
- Acknowledgement of Dr. Russell Coe at Suffolk County Community College who worked on an earlier version of this talk.

Mathematical Statement of the Collatz Conjecture

Let n be a positive integer. Then a sequence of integers can be defined as follows:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$

Collatz Function



Lothar Collatz (1910-1990)

(proposed in 1937)

The Collatz Conjecture asserts that this sequence always leads to the number 1 when one starts with any positive integer.

Illustration of How the Collatz Conjecture Works

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

Collatz Function

The Collatz function is an iterative function

$$f(f(n)) = f^2(n)$$

$$f(f(f(n))) = f^3(n)$$

We want, for any n,

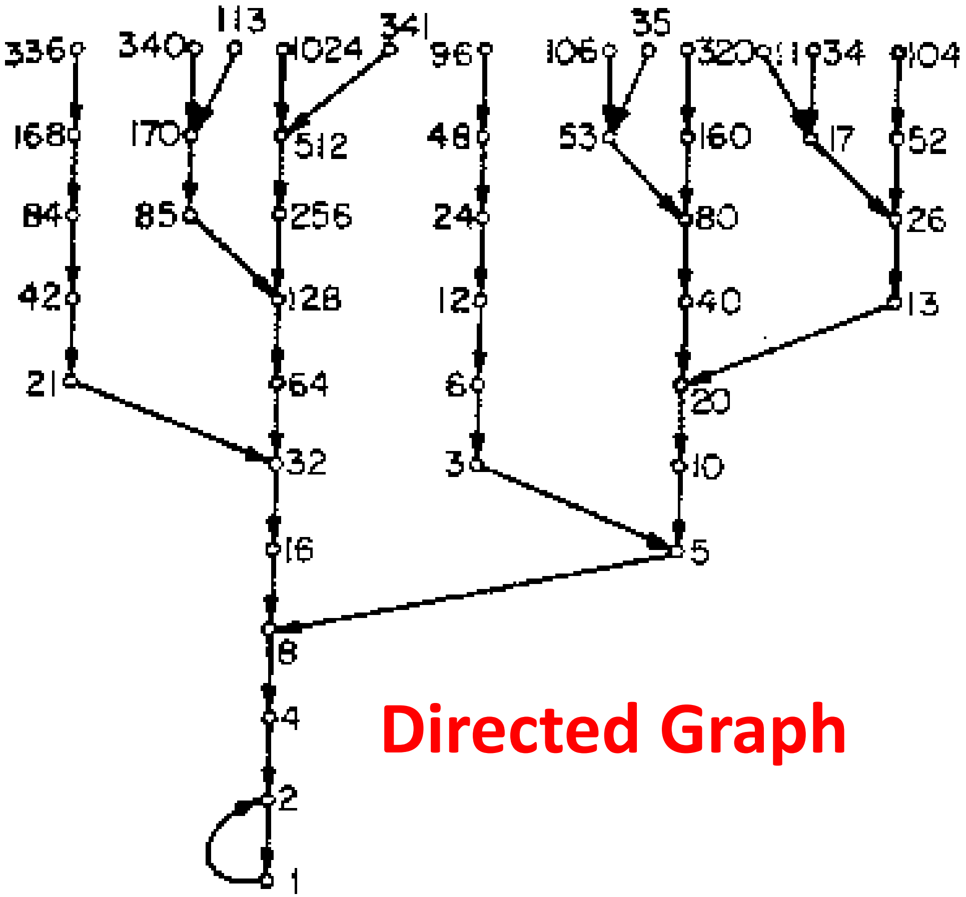
$$\lim_{k \to \infty} f^k(n) = 1$$

Each sequence of integers is an "orbit"

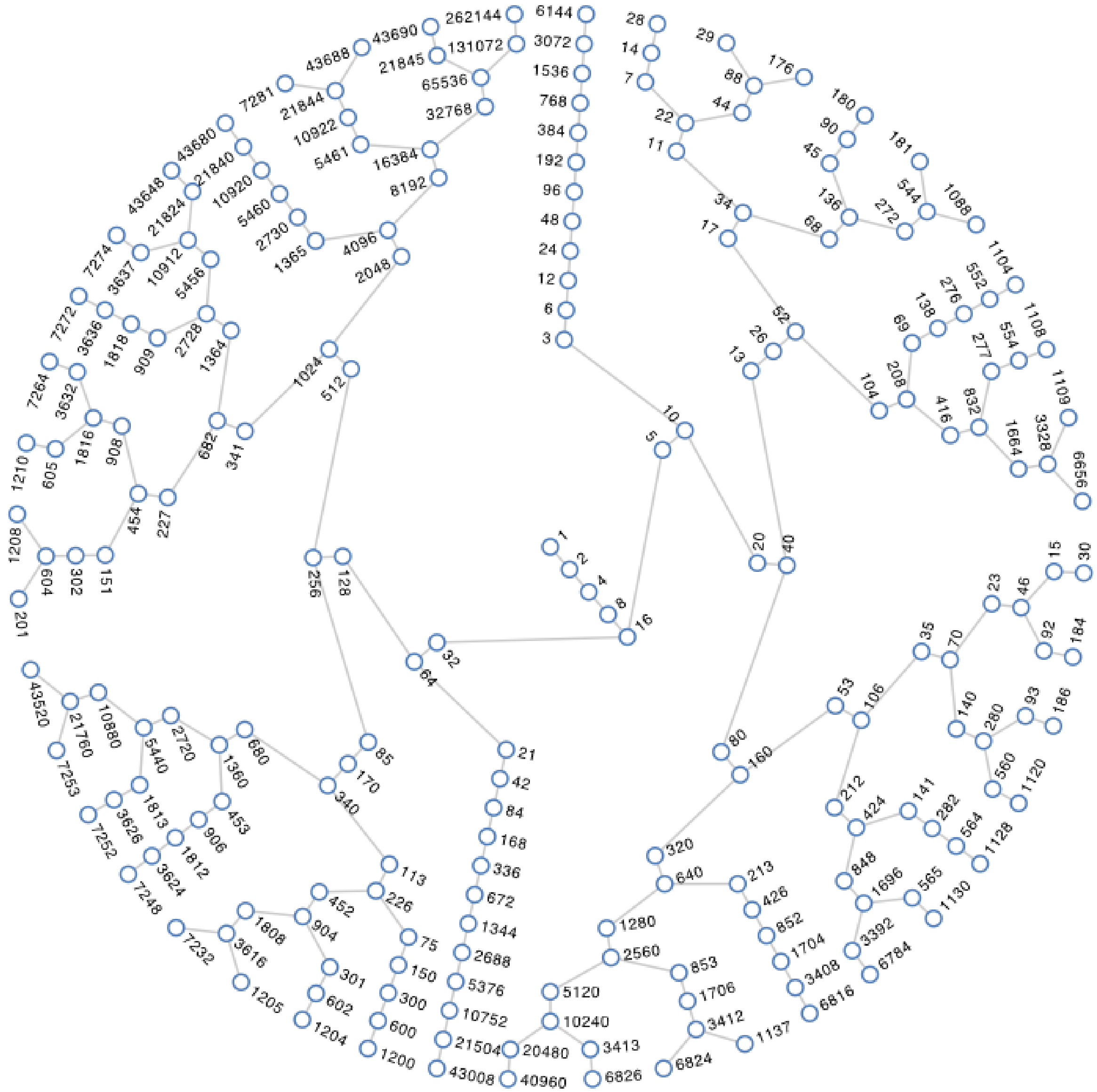
- 1
- 2 → 1
- 3 → 10 → 5 → 16 → 8 → 4 → 2 → 1
- 4 → 2 → 1
- 5 → 16 → 8 → 4 → 2 → 1
- 6 → 3 → 10 → 5 → 16 → 8 → 4 → 2 → 1

Illustration of How the Collatz Conjecture Works

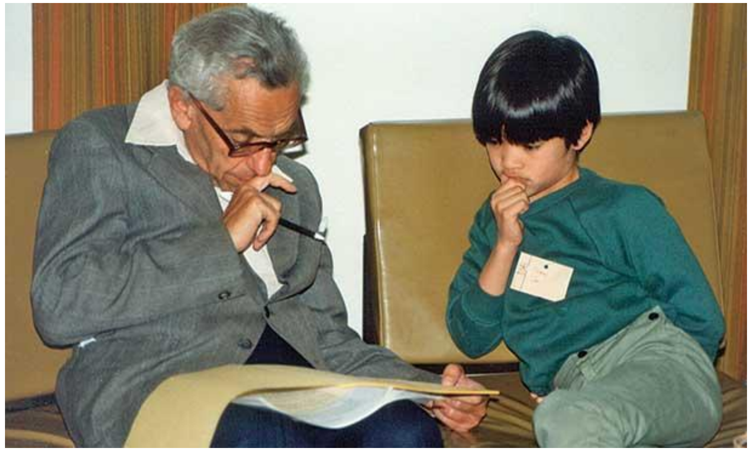
$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$



Directed Graph



To this date nobody has proven that the Collatz conjecture is true for all positive integers. Using computers, it has been verified to to work for every positive integer less than 2^{68} (approximately 2.9×10^{20}).



The prolific mathematician Paul Erdos said in 1983 that “Mathematics is not ready for such problems.”

Erdos offered a $500 prize for the proof or disproof of the Collatz Conjecture.

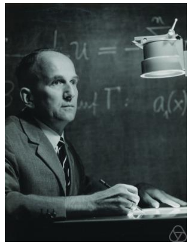
“This is an extraordinarily difficult problem, completely out of reach of present day mathematics.”

–Jeffrey Lagarias in 2010

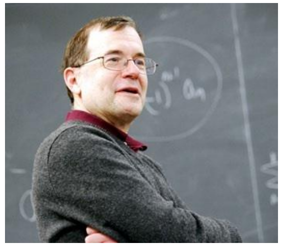
History of the Collatz Conjecture

- Lothar Collatz

  (**proposed in 1937**)



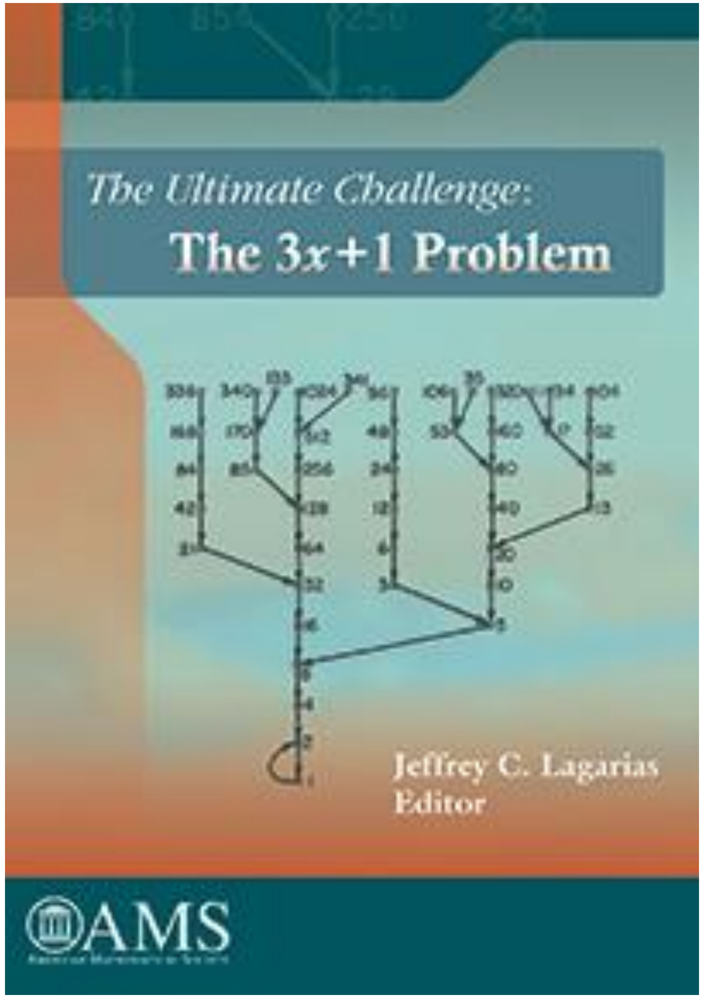
- Stanislaw Ulam (1950's)



- Jeffrey Lagarias

  (1967 to present)

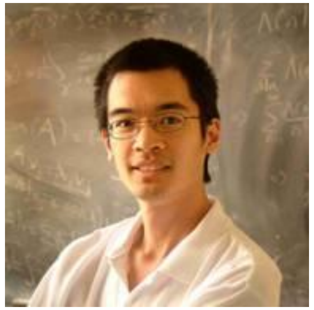
  The Ultimate Challenge: The 3x+1 Problem (2010 book)



- Riho Terras (1976) **"A Stopping-Time Problem on the Positive Integers" (1976 Paper)**

- Terence Tao (2006 Fields Medalist)

  "Almost all orbits of the Collatz map attain almost bounded values" (2019 Paper)



How Might We Try to Prove the Collatz Conjecture?

In "The Simple Math Problem We Still Can't Solve," in Quanta Magazine, Patrick Honner defined the following function to help explain the difficulty in proving the Collatz conjecture.

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd} \end{cases}$$

Nollatz function

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$

Collatz function

Nollatz sample orbits: **10, 5, 6, 3, 4, 2, 1, 2, 1** **11, 12, 6, 3, 4, 2, 1, 2, 1** **27, 28, 14, 7, 8, 4, 2, 1**

Nollatz Conjecture: Each orbit, regardless of the initial value, will reach 1.

- If n is even, the next number is $n/2 < n$. Thus, for half of the natural numbers n, **$g(n) < n$**.
- If n is odd, $g(n) = n + 1$ is even, and $g(g(n)) = (n+1)/2 = n/2 + \frac{1}{2}$, which is less than n if n is at least 3.
- Thus the orbit will always trend downwards.

- With the Collatz function, however, we don't know that a sequence will trend downward.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

Collatz function

- For even natural numbers n, the next number will be half of n.

- If n is odd, the next number will be 3n+1 which is even. The next number after that will be (3n+1)/2 = 1.5n + 0.5, which is larger than n.

- If (3n+1)/2 is even, then the next number will be (3n+1)/4 = 3n/4 +1/4, which is less than n if n > 1. For half of the odd numbers, the number will be less than n after three steps.

$5 \to 16 \to 8 \to 4 \to \ldots$

$7 \to 22 \to 11 \to 34 \to \ldots$

$9 \to 28 \to 14 \to 7 \to \ldots$

The Work of Riho Terras (1976) and others on the Collatz Conjecture

"A Stopping-Time Problem on the Positive Integers" – Riho Terras

In 1976, Terras showed that almost all initial values of n (meaning more than 99.99%) eventually became, through repeated application of the Collatz function, a value less than n.

For more than 99.99% of n, $n \rightarrow \ldots \rightarrow < n$

In 1979, Allouche showed that

For more than 99.99% of n, $n \rightarrow \ldots \rightarrow < n^{0.869}$

In 1994, Korec showed that

For more than 99.99% of n, $n \rightarrow \ldots \rightarrow < n^{0.793}$

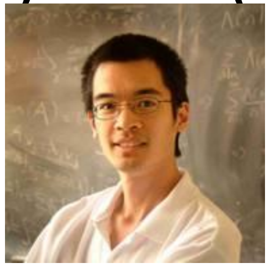
A Brief Sketch of What Terence Tao Did

"<u>Almost all orbits of the Collatz map attain almost bounded values</u>" (2019)

In 2019, Tao showed that almost all initial values of n (meaning more than 99.99%) eventually became, through repeated application of the Collatz function, a value less than any h(n) where h(n) is any function that grows to infinity.

For more than 99.99% of n, $n \rightarrow \ldots \rightarrow < h(n)$

For more than 99.99% of n, $n \rightarrow \ldots \rightarrow < \log(\log(\log(n)))$



- He wanted to find an appropriate sample of numbers and prove that if almost all (close to 100%) of the numbers in the sample end up at 1 or close to 1, then almost all numbers would end up at 1 or close to 1.

- One difficulty was trying to pick an appropriate sample of numbers that would represent all numbers.

- Different numbers have different properties (even, odd, multiples of 3, etc.) Some numbers differ from each other in only subtle ways.

- He needed to weight his sample to reflect the proportions of numbers with different properties. For example, if 90 percent of numbers shared a particular property, then he would want a sample where about 90% had that property.

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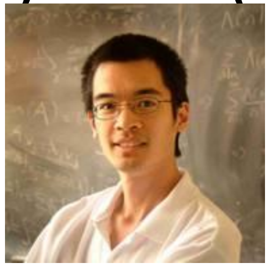

For more than 99.99% of n, $n \rightarrow \ldots \rightarrow < h(n)$

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A Brief Sketch of What Terence Tao Did

- The difficulty in choosing such a sample is that after choosing a sample and applying the conjecture a few times to each member of the sample, the resulting sample may no longer have the desired distribution of properties.

- He decided to avoid multiples of 3, as a Collatz sequence quickly loses all multiples of 3. (One more than a multiple of 3 has no factor of 3 in it. Dividing it by 2 wouldn’t change that. If a number is an even multiple of 3, after applying the function by the number of times the number is divisible by 2, it would be multiplied by 3 and increased by 1.)

- He also chose to use more numbers that when divided by 3 gave a remainder of 1 and not very many numbers that when divided by 3 gave a remainder of 2. A number with a remainder of 2 would eventually lead to a number with a remainder of 1.

A Brief Sketch of What Terence Tao Did

- **The sample he chose maintained its character after the Collatz algorithm is repeatedly applied.**
- **Using this sample he was able to prove that almost all (99.99% or more) of all starting values eventually reach a value below 200.**
- **Since it is known that each starting value up to 200 eventually leads to 1, the conjecture is proved to be true for at least 99.99% of all positive integers.**
- **Terence Tao's 2019 work on the Collatz Conjecture is a *tour de force* of mathematics.**
- **"It's a great advance in our knowledge of what's happening in this problem. It's certainly the best result in a long time." – Jeffrey Lagarias**
- **"You can get as close as you want to the Collatz Conjecture, but it's still out of reach" - Terence Tao**

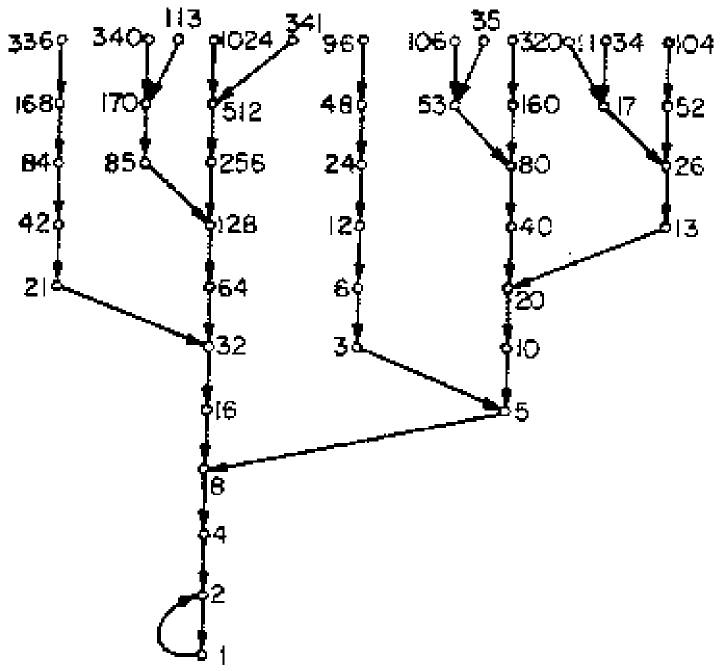
Why the Collatz Conjecture is Interesting for Our Students

- **<u>Suitable to discuss in a Discrete Math Class:</u>**

  - **Methods of Proof, i.e. Mathematical Induction**

  - **Examination of Cases is very important**

  - **Directed Graphs**



- **<u>Possible to discuss in a Math for Liberal Arts Class:</u>**

  - **Simple to state conjecture and illustrate with examples.**

  - **Some Math Problems are unsolved and perhaps unsolvable!**

Goldbach Conjecture
Collatz Conjecture
Reimann Hypothesis

References

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