Activating Your Classroom with the Five Practices

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Fired Up for Math
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All Shook Up

Nine people meet at a party. They all exchange handshakes. How many handshakes are exchanged?

Source: Shutterstock
Learning Goal

Students will be able to determine the number of handshakes in a variety of different ways including drawing pictures, solving a simpler problem, and /or counting (resolving along the way the dilemmas of “double counting” and “shaking hands with myself”).

(Depending on work present) Teacher can connect the (incorrect) rectangular solution \((9 \times 8)\) or the counting solution \(8+7+6+5+4+3+2+1\) to the triangular solution \(\frac{9\times8}{2}\), with the potential of generalizing to \(\frac{n(n-1)}{2} = \frac{9(9-1)}{2}\).
Section 1: Jan 23:

9 people meet at a party and exchange handshakes. How many handshakes occurred?

36 handshakes
876543210 = 36
Basic idea

- In class, provide cognitively demanding tasks.
  - Students work in groups (sometimes pairs)
  - Work is usually on white boards
- Students share their solutions with the class.
- Use student solutions/approaches to meet the goals of the lesson.
Five practices (Smith & Stein, 2011, 2018)

**Practice 0:** Set a Goal

**Practice 1:** Anticipation

**Practice 2:** Monitoring

**Practice 3:** Selecting

**Practice 4:** Sequencing

**Practice 5:** Connecting

**Legend**

Before Class

Planning to Teach, Students are Engaged (In Class)

Teaching is Visible (In Class)
Liberal Arts Mathematics or Quantitative Reasoning
Cows & Chickens

Mari and Jens went to their aunt’s farm. They noticed there were chickens in the cow pasture. Mari counted a total of 54 legs. Jens counted a total of 21 heads. How many chickens and how many cows were in the pasture?
54 legs, 21 heads

6 cows
15 chickens
54 legs, 21 heads

6 cows: 5 = 24 legs
15 chickens: 30 legs
24 + 30 = 54

6 cows
15 chickens

54 legs, 21 heads

Chicken: x
Cow: y

\[
\begin{align*}
6x + 15y &= 54 \\
6x + 15(21-x) &= 54 \\
6x + 315 - 15x &= 54 \\
-9x &= -261 \\
x &= 29
\end{align*}
\]

\[
\begin{align*}
y &= 21 - x \\
y &= 21 - 29 \\
y &= -8
\end{align*}
\]

\[
\begin{align*}
x &= 15 \text{ put into } 1 \\
x &= 15
\end{align*}
\]
Cutting & Stacking Paper

Take a sheet of paper, cut it in half (maybe fold it first), and then place one half on top of the other half to create a pile of paper with a height equal to the thickness of two sheets of paper. Take that pile, cut it in half (maybe fold it first), and place one half on top of the other. The resulting pile would have a height equal to four sheets of paper. Continue this process...

Question: How high would the pile be if you repeated this 50 times?

Source: Trinter & Garofalo, 2011
<table>
<thead>
<tr>
<th># of folds</th>
<th>thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.004</td>
</tr>
<tr>
<td>1</td>
<td>0.008</td>
</tr>
<tr>
<td>2</td>
<td>0.016</td>
</tr>
<tr>
<td>3</td>
<td>0.032 ( = 8 \times 0.004 )</td>
</tr>
<tr>
<td>4</td>
<td>0.064</td>
</tr>
<tr>
<td>5</td>
<td>0.128</td>
</tr>
<tr>
<td>6</td>
<td>0.256</td>
</tr>
<tr>
<td>7</td>
<td>0.512</td>
</tr>
<tr>
<td>8</td>
<td>1.024</td>
</tr>
</tbody>
</table>

Question: How high would the pile be if you repeated this 50 times?

<table>
<thead>
<tr>
<th># of folds</th>
<th>layers of paper</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2^1) \times 2</td>
</tr>
<tr>
<td>2</td>
<td>(2^2) \times 2</td>
</tr>
<tr>
<td>3</td>
<td>(2^3) \times 2</td>
</tr>
<tr>
<td>4</td>
<td>(2^4) \times 2</td>
</tr>
<tr>
<td>5</td>
<td>(2^5) \times 2</td>
</tr>
<tr>
<td>6</td>
<td>(2^6) \times 2</td>
</tr>
</tbody>
</table>

2\(^F\) = P

\[(2^{(F)})0.004 = T\]

\[2^{(60)} = 1.1258999 \times 10^{15}\]

\[\times 0.004 = \frac{T}{F} = 4.5036986 \times 10^{12}\]
<table>
<thead>
<tr>
<th>Folds</th>
<th>Layers</th>
<th>Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.004</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.008</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.016</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>0.032</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>0.064</td>
</tr>
</tbody>
</table>

\[ L \cdot 0.004 = T \]
\[ T = 2^F (0.004) \]

\[ T = 4 \text{ trillion in} \]
\[ T_{\text{in ft}} = 375 \text{ billion ft} \]
\[ \frac{T}{12} = T_{\text{in miles}} \]

71,079,539.573397 miles
Mathematics for Elementary School Teachers
The Bargain Task

Bargain

A shop advertised "Everything half price in our sale", but also now advertises that there is "An additional 15% off sale prices." Overall, this is equivalent to what reduction on the original prices?

Source: NRICH
50% discount followed by an additional 15% off sale prices.
50% \ a = b
15\ b

100 - (50\% \ 100) = 50
50 - (15\% \ 50) = 42.5
7.5
100 - 42.5 = \boxed{57.5}\%}

50% discount followed by an additional 15% off sale prices.

$50$

\[\frac{50}{5.5} = 9.09\]

\[50 + 7.5 = 57.5\%\]
Coupon vs. Discount

You have a coupon worth $18 off the purchase of a graphing calculator. At the same time, the calculator is offered with a discount of 15%, but no further discounts may be applied. For what tag price on the calculator do you pay the same amount for each discount?

Illustrative Mathematics (2016)
For what tag price on the calculator do you pay the same amount for each discount?

\[
\begin{align*}
X &= \text{calculator} \\
P &= \text{# of calculator} \\
P &= x - 18 \\
P &= x \cdot 0.85
\end{align*}
\]

The cost of the calculator is $120 at which you would pay the same amount for each discount.

\[
\begin{align*}
x - 18 &= x \cdot 0.85 \\
x - x &= -18 \\
-0.15x &= -18 \\
x &= \frac{-18}{-0.15} \\
x &= 120
\end{align*}
\]

\[
\begin{align*}
0.15x &= 18 \\
x &= \frac{18}{0.15} \\
x &= 120
\end{align*}
\]

\[
\begin{align*}
120 \\
& \quad \downarrow \\
1800 \\
& \quad \downarrow \\
15 \\
& \quad \downarrow \\
\text{with coupon} \\
0.15 \\
\end{align*}
\]
For what tag price on the calculator do you pay the same amount for each discount?

If $18$ is the same as $15\%$:

$$18 \div 15 = 1.2 \quad \text{(18 ÷ 15) x 100 = } x$$

$1.2$ is $1\%$ of the price. $1.2 \times 100 = 120$

The tag price is $\$120$.

$$\frac{18}{a} = \frac{5}{b} \quad (a \div b) \times 100 = x$$

$15\% = $18

$10\% = 12$

$5\% = 6$

$\$120$
For what tag price on the calculator do you pay the same amount for each discount?
Geometry
Let’s take a look at several student solutions to this problem:

Determine the sum $a + b + c + d + e + f$ of the angles in the 6-sided shape (hexagon) in Figure 10.35 without measuring. Explain your reasoning.

These three solutions are the most common ones among students:
This solution is a variation from the first three, but is related.

Student D

Do we need to explain where 360 degrees come from?
These two solutions are the most rare so they go after the first four solutions.

Student E

6x180-180-180=720 degrees

Student F

6x180=1080?
Why is this answer different?
Student G

Why does the formula work?

The sum of the angles in a \( n \)-sided polygon is given by 

\[
(n - 2) \times 180
\]

where \( n \) is the number of sides in a polygon: \( n = 6 \)

So, 

\[
(6 - 2) \times 180 = 4 \times 180 = 720
\]
Determine the sum $a + b + c + d + e + f$ of the angles in the 6-sided shape (hexagon) in Figure 10.35 without measuring. Explain your reasoning.

why does
$(n-2) \cdot 180^\circ$
work?

Student A

Student B

Student C
Student E

6x180 - 180 - 180 = 720 degrees

Why does \((n-2) \cdot 180^\circ\) work?

Student F

6x180 = 1080°?

Why is this answer different?
2. **Figure 12.14** shows the floor plan for a one-story house. Calculate the area of the floor of the house, explaining your reasoning.

![Floor plan of a house](image)

**Figure 12.14**  Floor plan of a house.
40 ft

\[
\begin{array}{c}
40 \times 24 \\
24 \times 16 \\
24 ft \\
40 ft
\end{array}
\]

40 ft

\[
\begin{array}{c}
24 \times 40 = 960 \\
10 \times 24 = 240 \\
32 \times 40 = 1280 \\
960 + 384 + 1280 = 2624 ft^2
\end{array}
\]

40 - 24 = 16
40

40 - 24 = 16

24 + 32 = 56

960 + 8960 + 768 = 9988
24 \times 40 = 960 \text{ ft}^2

40 + 24 = 64 \text{ ft}

64 \times 510 = 3,584 \text{ ft}^2

3,584 - 384 = 3,200 - 960

= 2,240 \text{ ft}^2
Overlap is $16 \times 16 = 256$

\[ \begin{align*}
1600 & \rightarrow \text{square 1 because } 40 \times 40 \\
1280 & \rightarrow \text{square 2 because } 40 \times 32 \\
\text{Add } 1600 & \text{ and subtract the overlap}
\end{align*} \]

\[ \begin{align*}
+1280 & \\
2,880 & - 256
\end{align*} \]

The area of the floor plan is $2,880 - 256 = 2,624$

$2,624 \text{ ft}^2$. I found this out by taking the 2 squares and finding the areas of the squares separately. Then since there was overlapping of the two squares I must subtract the area of the overlap from the 2 square areas. Therefore the total area is $2,624 \text{ ft}^2$. 
Calculus I
Problem

TRUE / FALSE

If \( f(x) < g(x) \) for all \( x \neq a \),
then \( \lim_{x \to a} f(x) < \lim_{x \to a} g(x) \).

Justify!!
TRUE/FALSE

If \( f(x) < g(x) \) for all \( x \neq a \),
then \( \lim_{x \to a} f(x) < \lim_{x \to a} g(x) \).

Justify!!
TRUE / FALSE

If \( f(x) < g(x) \) for all \( x \neq a \), then \( \lim_{x \to a} f(x) < \lim_{x \to a} g(x) \).

Justify!!
Problem

A machine is causing a particle to move along the $x$-axis so that its position at time $t$ is given by $x(t) = (t - 4)^2$, where $t$ is in seconds.

(a) What is the particle's velocity at $t = 2$? Interpret.

(b) The machine stops suddenly at $t = 3$, releasing the particle. As the particle continues, where will it be 5 seconds after the machine stops? Explain your thinking.


a. \( x'(t) = 2t - 8 \)
\[ x'(2) = 2(2) - 8 = -4 \]

b. \( x(0) = 1 \)
\[ x'(3) = -2 \text{ m/s} \]
\[ -2 \text{ m/s} \cdot 5 \text{ s} = -10 \text{ m}, \text{ moved} \]
\[ 1 - 10 = -9 \]

After 5 sec of the machine off, the particle would be at \( x = -9 \). The position of the particle at 3 seconds would be 1 unit. The particle's velocity is -2 m/s at 3 seconds. The particle continues at -2 m/s for 5 sec which means it moves -10 units. -10 units + 1 unit, the starting position, will put you at -9 units.
\[ x(t) = (t-4)^2 \]
\[ v = 2(t-4) \]
\[ v = 2t - 8 \]
\[ v = 2(2) - 8 \]
\[ v = -4 \]

**Diagram 1:**
- \( v = 2t - 8 \)
- \( v = 2(3) - 8 \)
- \( v = -2 = m \)
- \( 1 = -2(3) + b \)
- \( 1 = -6 + b \)
- \( 7 = b \)
- \( v = -20t + 7 \rightarrow 8 \sec(3 + 8) \)
- \( v = -2(8) + 7 \)
- \( v = -16 + 7 \)
- \( v = -9 \)

\[ b = (8, -9) \]

**Diagram 2:**
- 2 units/sec
- Initial position
- start: 3
- end: 7
- 9 units
- 3 units
- 7 units
- 5 units
- 3 units
- 1 unit
- 1 unit
Calculus II
Calculus II (Algebra of Inverses [Stewart])

Solve \( e^x + e^{-x} - 6 = 0 \).
Solve $e^x + e^{-x} - 6 = 0$.

<table>
<thead>
<tr>
<th>What happened...</th>
<th>Comment/Desired Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^x + e^{-x} = 6$</td>
<td>Take the log of both sides.</td>
</tr>
<tr>
<td>$e^{x-x} = 6$</td>
<td>Algebraic mistakes</td>
</tr>
<tr>
<td>$(e^x)^2 - 6e^x + 1 = 0$</td>
<td>Probably what I would have shown them (with direct instruction)</td>
</tr>
</tbody>
</table>
What students actually did...

\[ e^x + e^{-x} - 6 = 0 \]
\[ y = e^x, \quad e^x + \frac{1}{e^x} = 6 \]
\[ y + \frac{1}{y} = 6 \]
\[ y^2 + 1 = 6y \]
\[ y^2 - 6y + 1 = 0 \]
\[ c = 6 \pm \sqrt{36 - 4} \]
\[ y = \frac{6 \pm \sqrt{32}}{2} \]
\[ y = 5.828 \quad y = 1.171 \]
\[ e^x = 5.828, \quad e^x = 1.171 \]
\[ x = 1.762, \quad x = -1.762 \]
\[ e^x + e^{-x} - 6 = 0 \]
\[ e^x + e^{-x} = 6 \]
\[ (e^x + e^{-x})^2 = 36 \]
\[ e^{2x} + 2 + e^{-2x} = 36 \]
\[ e^{2x} + e^{-2x} = 34 \]
\[ e^{2x} - 2 + e^{-2x} = 32 \]
\[ (e^x - e^{-x})^2 = 32 \]
\[ e^x - e^{-x} = \pm 4 \sqrt{2} \]
\[ e^x + e^{-x} + e^x - e^{-x} = 6 \pm 4 \sqrt{2} \]
\[ 2e^x = 6 \pm 4 \sqrt{2} \]
\[ e^x = 3 \pm 2 \sqrt{2} \]
$x \approx \pm 1.7627$

$x = \ln \left( 3 \pm 2\sqrt{2} \right)$

$cosh(?) = 3$
Problem

Evaluate by any means:

\[ \int \cos^3 x \sin x \, dx \]
a) \[ \int \cos^3 x \sin x \, dx \]

\[- \int u^3 \, du \quad u = \cos x \]

\[- \int \sin x \, dx \quad du = -\sin x \, dx \]

\[- \frac{u^4}{4} + C \]

\[- \frac{1}{4} \cos^4 x + C \]

\[\int \cos^3 x \sin x \, dx \]

\[u = \cos x \]

\[du = -\sin x \, dx \]

\[-\int (\cos x)^3 (-\sin x \, dx) \]

\[= -\frac{\cos^4 x}{4} + C \]
\( \int \cos^0 x \sin x \, dx \)
\( = \) \( \int \cos^2 x \cos x \sin x \, dx \)
\( = \int \frac{1 - \sin^2 x}{\sin x} \cos x \sin x \, dx \)
\( \overset{u = \sin x}{\text{with } du = \cos x \, dx} \)
\( = \int (1 - u^2) \, du \)
\( = \int \frac{\sin^2 x}{2} + \frac{\sin^4 x}{4} + C \)

\( \int \cos^2 x \sin x \, dx \)
\( \overset{u = \cos x}{\text{with } du = -\sin x \, dx} \)
\( \int \frac{u}{2} \, du \)
\( = -\frac{1}{2} \frac{u^2}{2} \)
\( = -\frac{u^2}{4} + C \)
\( = -\frac{(\cos^2 x)^2}{4} + C \)
What is your goal in using a particular task?

**Continuum in Selecting/Sequencing:**

Least Sophisticated → Most Sophisticated
Informal → Formal
Incorrect Solutions → Valid Solutions
Most Common → Least Common
Miscellaneous....

1. Where can I find rich, thought-provoking tasks?
2. Where can I learn more about the 5 Practices?
Comments or Questions?

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End Matter
Tiling a Patio

Alfredo Gomez is designing patios. Each patio has a rectangular garden area in the center. Alfredo uses black tiles to represent the soil of the garden. Around each garden, he designs a border of white tiles. The pictures shown below show the three smallest patios that he can design with black tiles for the garden and white tiles for the border.

Patio 1

Patio 2

Patio 3

a. Draw patio 4 and patio 5. How many white tiles are in patio 4? Patio 5?

b. Make some observations about the patios that could help you describe larger patios.

c. Describe a method for finding the total number of white tiles needed for patio 50 (without constructing it).

d. Write a rule that could be used to determine the number of white tiles needed for any patio. Explain how your rule relates to the visual representation of the patio.

e. Write a different rule that could be used to determine the number of white tiles needed for any patio. Explain how your rule relates to the visual representation of the patio.

(Adapted from Cuevas and Yeatts [2005, pp. 18–22].)
1: \(9^2\)  
2: \(10^2\)  
3: \(12^2\)  
4: \(14\)

**x = white squares**  
**y = patio #**

**E. \(x = 2(y + 3)\)**

**C. \(x = 2y + 6\)**

- 1 black type  
  2 white -6
- 2 black squares  
  9 white -6
- 3 black  
  6 white

**e) \(3(p+2) - p\)** Finding total # of tiles and taking away the garden tiles.

b) garden = row, surrounded by white.

c) 50 black, 50 white on top, 50 white on bottom, and 3 white on each side. 106 white total.

d) \(2p + 6\): 2 white rows, same length as the black row, and 3 on each side.
b. There is always an endcap that has 3 tiles.
   There are always 3 rows.
   Every time it adds 2 more white tiles.

C. There are 2 white tiles for every black tile.
   So if there are 50 black tiles then there
   will be 100 white tiles plus the 6 on the
   ends. So it has 106 white tiles.

\[\text{C. } 2(x+2)+2 \text{ middle row has 2 white tiles.}\]

\[\text{d. } 6 + 2x = \]
\[\text{for every black tile there are 2 white tiles}\]
\[\text{Top row and bottom row is the 2, then every row has } (x+2) \text{ white tiles.}\]
The ratio of songs on Jessa’s phone to Tessie’s phone is 2 to 3. Tessie deletes half of her songs and now has 60 fewer songs than Jessa. How many songs does Jessa have?

Source: Engage NY
Jessa has 240 songs
<table>
<thead>
<tr>
<th>2:3</th>
<th>J J</th>
<th>T T T T</th>
</tr>
</thead>
</table>

| 2:1.5 | 60 60 60 60 | 60 60 60 |

-1/2

240 Songs

60 fewer means 60 is the difference between 2 and 1.5?
Problem

TRUE / FALSE

Given \( f(x) = \frac{x^2 - 4}{x - 2} \) and \( g(x) = x + 2 \), we can say the functions \( f \) and \( g \) are equal. Explain your reasoning!
The graphs won't be equal but they will be pretty close to looking the same. There needs to be restrictions on the good graph for them to be equal.

\[
\text{FALSE}
\]

\[
f(x) = \frac{x^3 - 4}{x - 2} = \frac{x + 2(x - 2)}{x - 2}
\]

\[
f(x) = x + 2 \quad x \neq 2
\]

false cause they are equal, but not at \(x = 2\)

\[
\frac{x^2 - 4}{x - 2} = \frac{x + 2}{x - 2}
\]

\[
\text{False} \quad @x = 2
\]

\[
\text{Undefined}
\]

\[
4
\]
Sample feedback (Calculus I)

- Classroom facilitated learning in a hands-on manner. Allowed students to test their knowledge as well as inspired critical thinking.

- I like how the professor put the class into groups to try and solve problems together with peers instead of constant presentation style instruction. I also like how the professor would be available to talk or discuss issues or questions outside of class.

- I liked how we used the whiteboards frequently during class. It gave me an opportunity to work on problems during class, which really helped me understand the concepts that were being taught in class.

- I liked that we worked in groups all the time. We were constantly working on problems instead of being lectured which I think is a good thing to keep people focused, plus the class periods went by super fast.

- I very much liked using whiteboards in class.

- Interactive activities during class, i.e. sharing thoughts and ideas towards solving Mathematical problems.

- It was a different style of teaching that I really enjoyed.

- I liked how hands on it was with group learning.
Problem

Produce an example of each below. A sequence that is...

(a) both bounded and monotonic.
(b) bounded but not monotonic.
(c) not bounded but monotonic.
(d) neither bounded nor monotonic.

Miscellaneous thought: We know a bounded and monotonic sequence converges. However, must a convergent sequence be both bounded and monotonic?
a. \( \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \) 

b. \( \left\{ (-1)^n \right\}_{n=1}^{\infty} \)

c. \( \left\{ n \right\}_{n=1}^{\infty} \)

d. \( \left\{ n \cos(n) \right\}_{n=1}^{\infty} \)

\[ \sum_{n=1}^{\infty} \frac{1}{n^3}, \frac{1}{3}, \frac{1}{5} \]

\( \left\{ \cos \pi n \right\} \)

-1, 1, -1, 1, ...

\( \left\{ (-2)^n \right\} \)

-2, 4, -8, 16, ...

\( \left\{ (-2)^n \right\} \)
Antiderivatives

If $f$ is an antiderivative of $g$, and $g$ is an antiderivative of $h$, then

(a) $h$ is an antiderivative of $f$.
(b) $h$ is the second derivative of $f$.
(c) $h$ is the derivative of $f''$.

Source: Cornell’s Good Questions
$f(x^2) + g(x^2) + h$

$f \Rightarrow g \Rightarrow h$

$f^{-1} g \Rightarrow h$

$f'' = h \Rightarrow f$
Warm Up

Sketch the graph of a function $y = f(x)$ that has each of the given characteristics below.

(a) $f(2) = f(4) = 0$

(b) $f'(x) < 0$ if $x < 3$

(c) $f'(3)$ does not exist

(d) $f'(x) > 0$ if $x > 3$

(e) $f''(x) < 0$ for $x \neq 3$
a) $f(2) = 0$
   $f(4) = 0$

b) $x < 3$
   decreasing

c) $f'(3)$ DNE

d) $x > 3$
   increasing

e) $x \neq 3$
   concave up
a) Zeros @ $x = 2$ and $4$
b) dec. left of 3
c) not differentiable @ $x = 3$
d) inc. right of 3
e) concave down where $x \neq 3$