Student-Led Number Talks in Calculus

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What is a classroom Number Talk?

“... five- to fifteen-minute conversations around purposefully crafted computation problems ... that provide the essential processes and habits of mind of doing math.” (Parrish, 2011)

Ruth Parker & Kathy Richardson

Cathy Humphreys & Ruth Parker

Sherry Parrish et al.
Mathematics is done mentally.

Children are encouraged to think of several strategies to solve the problem.

Special hand signals are used to indicate when you are still thinking, when you are ready to share, when you have multiple strategies...


Elicits conversation. In doing so, NTs blur the line between procedural fluency and conceptual understanding.
Number Talk Hand Signals

I’m thinking.

I have a strategy.

I agree.

I have more than one strategy.
18 \times 5
$18 \times 5$

Source: Stanford Online’s “How To Learn Math for Teachers and Parents”: Number Talks
Which number doesn’t belong?

9   16   25   43
45% of 60
Number Talk String

\[ \frac{1}{2} \text{ of } 12 \]
Number Talk String

\[
\frac{1}{2} \quad \text{of} \quad 12
\]

\[
\frac{1}{4} \quad \text{of} \quad 12
\]
Number Talk String

\[
\frac{1}{2} \text{ of } 12
\]

\[
\frac{1}{4} \text{ of } 12
\]

\[
\frac{3}{4} \text{ of } 12
\]
Two Modifications

(1) Extending this to university mathematics:

Calculus II

Topic: Techniques of Integration

(2) Let Students Lead the “Number Talk.”
Integral Types

(1) **STRATEGY**: In how many ways can you approach this?

(2) **RECONCILIATION**: Are these (different-looking answers) equivalent?

(3) **TRANSFORMATION**: What can I do to make this solvable?
STRATEGY
Jim’s Problem

\[ \int \cos x \sin x \, dx \]
\[
\int \cos x \cdot \sin x \, dx = \int u \, dv = \int \sin x \, dv = \int \cos x \, dx
\]

\[
= \sin^2 x - \int \sin x \cos x \, dx + \int \cos x \sin x \, dx
\]

\[
2 \int \cos x \sin x \, dx = \sin^2 x
\]

\[
\int \cos x \sin x \, dx = \frac{1}{2} \sin^3 x + C
\]
\[
\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C
\]
\[
\int \sin x \cos x \, dx
\]

\[
= \int u \, dv
\]

\[
v = \sin x
\]

\[
dv = \cos x \, dx
\]

\[
\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C
\]

\[
v = \cos x
\]

\[
dv = -\sin x \, dx
\]

\[
\int \sin x \cos x \, dx = \frac{1}{2} \cos^2 x + C
\]

\[
= \frac{1}{2} (1 - \sin^2 x) + C
\]

\[
= -\frac{1}{2} + \frac{1}{2} \sin^2 x + C
\]
George’s Problem

\[ \int \frac{1}{x^4 - x} \, dx \]
\[
\int \frac{1}{x^4 - x} \, dx = \int \frac{1}{x(x^3 - 1)} \, dx = \int \frac{1 + x^2 - x^3}{x(x^3 - 1)} \, dx = \int \frac{-1 + x^2 - x}{x(x^3 - 1)} \, dx = \left[ -\int \frac{x^2 - 1}{x(x^3 - 1)} \, dx + \frac{x^3 - x^2}{x(x^3 - 1)} \right] \, dx
\]

\[
= \left[ -\int \frac{1}{x} \, dx + \int \frac{x^2}{x(x^3 - 1)} \, dx \right] \quad u = x^3 - 1, \quad du = 3x^2 \, dx
\]

\[
= -\ln x + C \quad = \frac{1}{3} \int \frac{3x^2}{x^3 - 1} \, dx = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln u + C
\]

\[
= \frac{1}{3} \ln |x^3 - 1| + C
\]

\[
\Rightarrow \quad -\ln x + \frac{1}{3} \ln |x^3 - 1| + C
\]
\[ \int \frac{1}{x^4 - x} \, dx = \int \frac{1}{x^4(1 - x^{-3})} \, dx \quad u = 1 - x^{-3} \quad du = 3x^{-4} \, dx \]

\[ = \frac{1}{3} \int \frac{3x^{-4}}{1 - x^{-3}} \, dx = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln |u| + C \]

\[ = \frac{1}{3} \ln |1 - x^{-3}| + C \]
Ming’s Problem

\[ \int \frac{x}{\sqrt{x^2 - 4}} \, dx \]
Way 1

Based on graph

\[ \frac{x}{z} = \sec \theta \]

\[ x = 2 \sec \theta \]

\[ dx = 2 \tan \theta \sec \theta \]

Based on graph

\[ \tan \theta = \frac{\sqrt{x^2 - 4}}{2} \]

\[ 2 \tan \theta = \sqrt{x^2 - 4} \]

\[ \int \frac{x}{x^2 - 4} \, dx = \int \frac{2 \sec \theta}{2 \tan \theta} \, d\theta = \int 2 \sec \theta \tan \theta \, d\theta \]

\[ = 2 \tan \theta + C = \frac{\sqrt{x^2 - 4}}{2} + C \]
Way 1

\[ \frac{x}{y} = \frac{\sqrt{x^2 - 4}}{y} \]

Based on graph

\[ \frac{x}{2} = \sec \theta \]

\[ x = 2 \sec \theta \]

\[ \frac{dx}{2} = \sec \theta \tan \theta \, d\theta \]

Based on graph

\[ \tan \theta = \frac{\sqrt{x^2 - 4}}{2} \]

\[ 2 \tan \theta = \sqrt{x^2 - 4} \]

\[ \int \frac{x}{\sqrt{x^2 - 4}} \, dx = 2 \sec \theta \tan \theta \, d\theta = 2 \sec^2 \theta \, d\theta \]

\[ = 2 \tan \theta + C = \sqrt{x^2 - 4} + C \]

Method 2

Make \( u = x^2 - 4 \)

So \( du = 2x \, dx \)

So \( \int \frac{x}{\sqrt{x^2 - 4}} \, dx = \frac{1}{2} \int \frac{2x}{\sqrt{x^2 - 4}} \, dx \)

\[ = \frac{1}{2} \int \frac{du}{\sqrt{u^2}} \]

\[ = \frac{1}{2} \text{ln} |u| - \frac{1}{2} du = \frac{1}{2} (x^2 - 4)^{\frac{1}{2}} + C = \]

\[ \sqrt{x^2 - 4} + C \]
Method 3
\[
\int \frac{x}{\sqrt{n^2 - x^2}} \, dx = \int \frac{x}{\sqrt{n^2 - x^2}} \, dx = \int \frac{\sqrt{n^2 + x^2} + \sqrt{n^2 - x^2}}{\sqrt{n^2 + x^2}} \, dx
\]
\[
= \int \frac{\sqrt{n^2 + x^2}}{\sqrt{n^2 + x^2}} \, dx + \int \frac{\sqrt{n^2 - x^2}}{\sqrt{n^2 + x^2}} \, dx
\]
\[
= \int \frac{1}{2n^2 x} \, dx + \int \frac{n^2 - x^2}{2n^2 x^2} \, dx
\]

make \( U = \sqrt{n^2 - x^2} \)
\[
dU = -\frac{x}{\sqrt{n^2 - x^2}} \, dx
\]
\[
dv = \frac{1}{2n^2 x} \, dx
\]
\[
v = \frac{1}{2n^2 x}
\]

so
\[
\int \frac{\sqrt{n^2 - x^2}}{2n^2 x^2} \, dx = \frac{\sqrt{n^2 - x^2}}{2n^2 x} - \frac{1}{2n^2 x}
\]

\[
\int \frac{\sqrt{n^2 + x^2}}{2n^2 x} \, dx = \frac{\sqrt{n^2 + x^2}}{2n^2 x} - \frac{1}{2n^2 x}
\]

\[
\int \frac{\sqrt{n^2 + x^2}}{\sqrt{n^2 - x^2}} \, dx = \sqrt{n^2 + x^2} - \sqrt{n^2 - x^2} - \frac{1}{\sqrt{n^2 - x^2}} \, dx
\]

\[
= \sqrt{n^2 + x^2} + \int \frac{n^2 - x^2}{\sqrt{n^2 + x^2}} \, dx
\]

\[
= \sqrt{n^2 + x^2} + \frac{n^2}{2} \int \frac{\sqrt{n^2 + x^2}}{\sqrt{n^2 - x^2}} \, dx
\]

\[
= \frac{n^2}{2} \sqrt{n^2 + x^2} - \frac{1}{2} \sqrt{n^2 - x^2} - \frac{1}{\sqrt{n^2 - x^2}} + C
\]

\[
= \frac{n^2}{2} \sqrt{n^2 + x^2} - \frac{1}{2} \sqrt{n^2 - x^2} + C
\]
RECONCILIATION OF THE ANSWER
Edward’s Problem

\[ \int \tan x \sec^2 x \, dx \]
\[ \int \tan x \sec^2 x \, dx \]
\[ = \int u \, du \]
\[ = \frac{1}{2} u^2 + C \]
\[ = \frac{1}{2} tan^2 x + C \]
\[ \int \tan x \sec^2 x \, dx \]
\[ = \frac{1}{2} u^2 + C \]
\[ = \frac{1}{2} \tan^2 x + C \]

\[ \int \sec x \, dx \]
\[ u = \tan x \]
\[ du = \sec^2 x \, dx \]

\[ \int \sec x \tan x \sec x \, dx \]
\[ u = \sec x \]
\[ du = \tan x \sec x \, dx \]
\[
\int \tan x \sec^2 x \, dx \\
= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x} \, dx \\
= \int \frac{\sin x}{\cos^3 x} \, dx \\
= \int \frac{-1}{u^3} \, du \\
= -\int u^{-3} \, du \\
= -\left( -\frac{1}{2} u^{-2} \right) + C \\
= \frac{1}{2} (\cos x)^{-2} + C \\
= \frac{1}{2} \frac{1}{(\cos x)^2} + C \\
= \frac{1}{2} (\sec x)^2 + C
\]
Neva’s Problem

\[ \int \frac{x}{x^2 + 9} \, dx \]
\[ \int \frac{x}{x^2+9} \, dx \]

\[ u = x^2 + 9 \]
\[ du = 2x \, dx \]
\[ \frac{1}{2} \int \frac{du}{u} \]
\[ \frac{1}{2} \ln |u| + C \]
\[ \frac{1}{2} \ln |x^2 + 9| + C \]
\[
\int \frac{x}{x^2 + q} \, dx = \int \frac{x}{(\sqrt{x^2 + q})^2} \, dx \\
\begin{align*}
&= \int \frac{3\tan \theta}{(3\sec \theta)^2} \, d\theta \\
&= \int \frac{3\tan \theta}{9\sec^2 \theta} \, d\theta \\
&= \frac{1}{3} \int \frac{\tan \theta}{\sec \theta} \, d\theta \\
&= \frac{1}{3} \ln \left| \cos \theta \right| + C \\
&= \frac{1}{3} \ln \left| \frac{3}{\sqrt{x^2 + q}} \right| + C \\
&= \frac{1}{3} \ln \left| \frac{3}{\sqrt{x^2 + q}} \right| + C \\
&= \frac{1}{3} \ln \left| x^2 + q \right| + C
\end{align*}
\]
TRANSFORMATION
Tammy’s Problem
\[ \int e^{\sqrt{x}} \, dx \]
Method 1: $u$-Substitution

\[
\int e^{\sqrt{x}} \, dx \quad u = \sqrt{x} \quad du = \frac{1}{2\sqrt{x}} \, dx \Rightarrow du = \frac{1}{2u} \, dx, \quad dx = 2udu
\]

2. $\int ue^udu$ Now integrate by parts.

\[
v = u \quad w = e^u \quad \int vdw = vw - \int wdv
\]
\[
dv = 1 \, du \quad dw = e^u \, du
\]

\[2\int u e^u \, du = 2\left(ue^u - \int e^u \, du\right) = 2ue^u - 2e^u + C
\]

\[
\int e^{\sqrt{x}} \, dx = 2\left(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right), \quad \int e^{\sqrt{x}} \, dx = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C
\]
Method 2: $u$-substitution (part II)

\[ \int e^{\sqrt{x}} \, dx \quad u = e^{\sqrt{x}} \quad du = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \, dx, \quad dx = \frac{2\sqrt{x}}{e^{\sqrt{x}}} \, du = \frac{2\ln(u)}{u} \, du \]

\[ 2 \int u \ln(u) \, du = 2 \int \ln(u) \, du \quad \text{Integrate by parts:} \quad V = \ln(u) \quad W = u \]

\[ \int v \, dw = vw - \int w \, dv \quad dv = \frac{1}{u} \, du \quad dw = du \]

\[ \ln(u) = u \ln(u) - \int \frac{u}{u} \, du \]

\[ 2 \int \ln(u) \, du = 2(u \ln(u) - u) + C, \quad \int e^{\sqrt{x}} \, dx = 2(e^{\sqrt{x}} - e^{\frac{x}{2}}) + C \]

\[ \int e^{\sqrt{x}} \, dx = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C \]
Method 3: Integration By Parts

\[ \int e^{\sqrt{x}} \, dx \]

\[ u = e^{\sqrt{x}} \quad v = x \]

\[ du = e^{\sqrt{x}} \frac{1}{2\sqrt{x}} \, dx \quad dv = dx \]

\[ \int u \, dv = uv - \int v \, du \]

\[ \int e^{\sqrt{x}} \, dx = xe^{\sqrt{x}} + \int \frac{x e^{\sqrt{x}}}{2\sqrt{x}} \, dx \rightarrow \text{Much Harder!} \]

\[ u = 1 \quad v = \int e^{\sqrt{x}} \, dx \rightarrow \text{Doesn't Help Either} \]

\[ du = dx \quad dv = e^{\sqrt{x}} \, dx \]
Carol’s Problem

\[ \int x^3 e^{\sqrt{x}} \, dx \]
\[ \int_{3}^{\sqrt{3}} x e^x \, dx \quad u = x^3 \quad dv = e^{x} \, dx \]

\[ du = 3x^2 \, dx \quad v = e^x \]
\[ \int x^3 e^{\sqrt{x}} \, dx \]

\[ u = x^3 \quad dv = e^{\sqrt{x}} \, dx \]

\[ du = 3x^2 \, dx \quad v = \int e^{\sqrt{x}} \, dx \]

\[ \int x e^x \, dx \]

\[ u = x \quad dv = e^x \, dx \]

\[ du = 1 \, dx \quad v = e^x \]
\[ \int_{1}^{3} x^3 e^{x^2} \, dx \]

\[ U = \sqrt{x}, \quad U^2 = x \]

\[ du = \frac{1}{2 \sqrt{x}} \, dx \quad U^2 = \frac{x}{2} \]

\[ 2u \, du = \frac{1}{2} \, dx \]

\[ \int x^3 e^{x^2} \, dx \]

\[ \int u^4 e^u \, 2u' \, du \]

\[ 2 \int u^7 e^u \, du \]
Limits, Derivatives, Series, etc.
Amy’s Problem

Find the derivative

of \( y = (x^3 + 7)^2 \).
\[ y = (x^3 + 7)^2 \]
\[ y = (x^3 + 7)(x^3 + 7) \]
\[ y = x^6 + 7x^3 + 7x^3 + 49 \]
\[ y = x^6 + 14x^3 + 49 \]
\[ y' = 6x^5 + 42x \]
\[ y = (x^3 + 7)^2 \]
\[ y = (x^3 + 7) \cdot (x^3 + 7) \]
\[ y = x^6 + 7x^3 + 7x^3 + 49 \]
\[ y = x^6 + 14x^3 + 49 \]
\[ y' = 6x^5 + 42x^2 \]

\[ y = (x^3 + 7)^2 \]
\[ y = u^2 \]
\[ u = x^3 + 7 \]
\[ y' = 2u \cdot \frac{du}{dx} \]
\[ y' = 2(x^3 + 7) (3x^2) \]
\[ y' = 6x^2(x^3 + 7) \]
\[ y' = 6x^5 + 42x^2 \]
\[ y = (x^3 + 7)^2 \quad \text{Chain Rule} \]
\[ y' = 2(x^3 + 7) \cdot (3x^2) \]
\[ y' = 6x^2(x^3 + 7) \]
\[ y' = 6x^5 + 42x^2 \]
\[ y = (x^3 + 7)^2 \quad \text{Chain Rule} \]
\[ y' = 2(x^3 + 7) \cdot (3x^2) \]
\[ y' = 6x^2 (x^3 + 7) \]
\[ \boxed{y' = 6x^5 + 42x^2} \]
Liam’s Problem

Does \( \sum_{n=1}^{\infty} \pi^{-n} \) converge or diverge?
Abs Convergent

\[
\lim_{n \to \infty} \frac{1}{n^2} = 0
\]

\[
\lim_{n \to \infty} \sqrt{n} = \infty
\]
\[ \sum_{n=1}^{\infty} \pi^{-n} \]
\[ \prod_{n=1}^{\infty} \pi^{-n} = \prod_{n=1}^{\infty} \frac{1}{\pi^n} \]

\[ \lim_{n \to \infty} n \sqrt[3]{\frac{1}{n}} \]
\[ \lim_{n \to \infty} \left( \frac{1}{n^3} \right) \]
\[ \left( \frac{1}{3} \right)^n \geq \left( \frac{1}{\pi} \right)^n \]

Converges

Abs Convergent

DCT
\[
\sum_{n=1}^{\infty} \pi^{-n}
\]

\[
\sum_{n=1}^{\infty} \left( \frac{1}{\pi} \right)^n
\]

\[
\frac{a}{1-r} = 1, \quad r = \frac{1}{\pi}
\]

\[
\frac{1}{1-\frac{1}{\pi}}
\]

-1 < \frac{1}{\pi} < 1

CONVERGES
Personal Reflections (Calculus II—Integration)

Remarks from students:
Not needing to study integration techniques as much (e.g., “I knew all the methods from the Integration Talks!”)
“Can I have another problem to do?”
“Is the rest of Calculus II this much fun?”
“I find it interesting that just 5-6 methods help us with so many problems.”
“I’m going to use this idea in my class.” (secondary ed major)
Personal Reflections (Calculus II—Integration)

What I learned:

Students love integration by parts! Sometimes, the “standard” approach was not the first method suggested. In some cases, it never came up at all.
For more information...

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