Definition of the Derivative.

Find the slope of the tangent to the graph of \( y = f(x) \) at the point where \( x = 2 \).

Questions?
1. What is a tangent?
2. How can we compute slope?
3. Can we find approximate solution?

\[
(2, f(2))
\]
\[
(2 + h, f(2 + h))
\]
\[
m = \frac{\Delta y}{\Delta x} = \frac{f(2 + h) - f(2)}{(2 + h) - 2}
\]

\[
\text{Msecant} = \frac{\Delta y}{\Delta x} = \frac{f(2 + h) - f(2)}{(2 + h) - 2}
\]

\[
\text{M_tangent} = \lim_{h \to 0} m_{secant} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}
\]

Notation:
\( f'(2) \) means the slope of the tangent to the graph of \( y = f(x) \) at the point where \( x = 2 \).

So \( f'(2) = M_{tangent} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} \)

Example: Let \( f(x) = 3^x \). Approximate \( f'(2) \)

\[
(2, 9)
\]
\[
(2, 9.0451)
\]
\[
\text{Msec} = \frac{\Delta y}{\Delta x} = \frac{10.0451 - 9}{1} = 10.45
\]

Question: Does \( f'(2) = 10.45 \)?

Question: What does \( f'(x) \) mean?

Question: What type of graph would not have the slope of a tangent at \( x = 2 \)?
Derivatives of Inverse Functions.

Let \( f(x) = \tan^{-1} 2x \). Find \( f'(x) \).

If \( y = \tan^{-1} 2x \), then \( \tan y = 2x \).

But \( \tan \theta = \frac{\text{opp}}{\text{adj}} \) so \( \frac{c}{l} = \frac{2x}{l} \).

\[ 1^2 + (2x)^2 = c^2 \Rightarrow 1 + 4x^2 = c^2 \Rightarrow \sqrt{1 + 4x^2} = c. \]

\[ \sqrt{1 + 4x^2} \]

\[ \frac{2x}{l} \]

\[ 1 \]

\[ \tan y = 2x \]

\[ \frac{d}{dx} \tan y = \frac{d}{dx} 2x \]

\[ \sec^2 y \frac{dy}{dx} = 2 \]

\[ \frac{dy}{dx} = \frac{2}{\sec^2 y} \]

\[ \frac{dy}{dx} = \frac{2}{1 + 4x^2} \]

\[ \frac{2}{1 + 4x^2} \]

So when \( f(x) = \tan^{-1} 2x \)

\[ f'(x) = \frac{2}{1 + 4x^2} \]

Let \( y = f^{-1}(x) \). Find \( y' \).

\( f(y) = x \Rightarrow \frac{dx}{dy} f(y) = x \Rightarrow f'(y) \frac{dy}{dx} = 1 \)

\[ \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} \Rightarrow \frac{dy}{d\theta} = \frac{1}{f'(f^{-1}(x))} \]

Arc-Length

Find the arc length of \( y = f(x) \) on \([a, b]\).

From Pythagoras we know \((\Delta x)^2 + (\Delta y)^2 = (\Delta \theta)^2\).

So \( \Delta A = \sqrt{(\Delta x)^2 + (\Delta y)^2} \)

Now break up the graph into many small parts.

So total arc length \( \Delta A \)

\[ \Delta A = \sqrt{(\Delta x)^2 + (\Delta y)^2} \]

\[ \Delta A = \sqrt{(\Delta x)^2 + (\Delta y)^2} \Delta x \]

Shrink \( \Delta x \) and \( \Delta y \) to \( 0 \).

\[ \Delta A = \int \Delta \]

\[ \Delta A = \int \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]
Taylor and Maclaurin Series

Find a Maclaurin series for \( f(x) = e^{2x} \)

Goal: Find a polynomial \( p(x) \) that approximates \( f(x) \) as \( x \to 0 \)

\[
p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots
\]

\[
f(x) = e^{2x}
\]

We want \( p(0) = f(0) \), \( p'(0) = f'(0) \), \( p''(0) = f''(0) \), etc.

\[
p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots
\]

\[
p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots
\]

\[
p''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots
\]

\[
p'''(x) = 6a_3 + 24a_4 x + 60a_5 x^2 + \cdots
\]

\[
p^{IV}(x) = 24a_4 + 120a_5 x + \cdots
\]

\[
p(v)(x) = 120a_5 + x^3 + x^2 + x^3 + \cdots
\]

\[
p(0) = a_0 \quad p'(0) = a_1 \quad p''(0) = 2a_2 \quad p'''(0) = 6a_3
\]

\[
p^{IV}(0) = 24a_4 \quad p(v)(0) = 120a_5 \quad p^{(w)}(0) = ?
\]

\[
f(x) = e^{2x} \quad f(0) = 1
\]

\[
f'(x) = 2e^{2x} \quad f'(0) = 2
\]

\[
f''(x) = 4e^{2x} \quad f''(0) = 4
\]

\[
f'''(x) = 8e^{2x} \quad f'''(0) = 8
\]

\[
f^{IV}(x) = 16e^{2x} \quad f^{IV}(0) = 16
\]

\[
f^{(v)}(x) = 32e^{2x} \quad f^{(v)}(0) = 32
\]

\[
p(0) = f(0) \quad p^{(w)}(0) = f^{(w)}(0)
\]

\[
a_0 = 1 \quad 24a_4 = 16
\]

\[
a_1 = 2 \quad a_4 = 4 \quad \frac{8}{24} = \frac{2}{3}
\]

\[
a_2 = 2 \quad 120a_5 = 32
\]

\[
a_3 = \frac{8}{6} = \frac{4}{3}
\]

\[
a_4 = \frac{64}{240} = \frac{4}{15}
\]

\[
a_5 = \frac{512}{1200} = \frac{4}{15}
\]

\[
\frac{4}{15} x^5 + \frac{4}{45} x^6 + \cdots
\]

Is there a pattern?

\[
p^{(n)}(0) = N! a_n
\]

We need \( p^{(n)}(0) = f^{(n)}(0) \) so

\[
N! a_n = f^{(n)}(0)
\]

\[
a_n = \frac{f^{(n)}(0)}{N!}
\]
Surface Area

Find the area of the surface \( z = f(x, y) \) above the region \( R \).

\[ (x_0, y_0, f(x_0, y_0)) \]

"Glue" a small piece of "papier-mâché" that represents the tangent plane to \( (x_0, y_0) \) at the point \( (x_0, y_0, f(x_0, y_0)) \).

\[ \langle \Delta x, 0, f_x \Delta x \rangle = \mathbf{u} \]

Another vector is

\[ \langle 0, \Delta y, f_y \Delta y \rangle = \mathbf{v} \]

What is the area of that piece of papier-mâché?

Answer: \( \| \mathbf{u} \times \mathbf{v} \| \)

\[ \mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} i & j & k \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{bmatrix} \]

\[ \mathbf{u} \times \mathbf{v} = \begin{bmatrix} -f_x \Delta x \Delta y, -f_y \Delta x \Delta y, \Delta x \Delta y \end{bmatrix} \]

\[ \| \mathbf{u} \times \mathbf{v} \| = \sqrt{f_x^2 + f_y^2 + 1} \Delta x \Delta y \]

The total surface area is the sum of all the pieces of papier-mâché over the region \( R \).

\[ \mathcal{S} \equiv \sqrt{f_x^2 + f_y^2 + 1} \Delta x \Delta y \]

Shrink \( \Delta x \) and \( \Delta y \) to zero, and get

\[ \mathcal{S} = \iint_{R} \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy \]