Lessons from the History of Mathematics

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NCTM Annual Meeting
Washington, DC
April 23, 2009

These slides are available at
www.macalester.edu/~bressoud/talks
“The task of the educator is to make the child’s spirit pass again where its forefathers have gone, moving rapidly through certain stages but suppressing none of them. In this regard, the history of science must be our guide.”

Henri Poincaré

1854–1912
Trigonometry

The problems:

• The emphasis on trigonometric functions as ratios of sides of various right triangles often leads students to view sine, cosine, etc. as functions of a triangle rather than an angle.

• Students are often confused about how angle measure is defined and have trouble working with radian measure of angles.

• Students have trouble making the transition to trigonometric functions as periodic functions with continuously varying input and output.

Common student and teacher difficulties with trigonometric functions have been studied by Pat Thompson and Kevin Moore at Arizona State University.
Old Babylonian Empire, 2000–1600 BCE
Plimpton 322 (Columbia University), table of Pythagorean triples, circa 1700 BCE, about 9 by 13 cm

Mathematical Association of America, 2004
Simple Pythagorean triples:

\[ 3^2 + 4^2 = 5^2 \]
\[ 5^2 + 12^2 = 13^2 \]
\[ 8^2 + 15^2 = 17^2 \]
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The difference between the square of side $A+B$ and the square of side $B-A$ is four rectangles of size $AB$.

$$(A + B)^2 - (B - A)^2 = 4AB$$

If $AB = \frac{1}{4}$, then $(A + B)^2$ and $(A - B)^2$ differ by 1.

$$A = \frac{6}{5}, \quad B = \frac{5}{24}$$

$$\frac{6}{5} \times \frac{5}{24} = \frac{1}{4}$$

$$\frac{6}{5} - \frac{5}{24} = \frac{119}{120}, \quad \frac{6}{5} + \frac{5}{24} = \frac{169}{120}$$

$$\left(\frac{169}{120}\right)^2 - \left(\frac{119}{120}\right)^2 = 1$$

$$169^2 = 119^2 + 120^2$$
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<td>9/10, 5/18</td>
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Hipparchus of Rhodes

*Circa* 190–120 BC

Princeton University Press, 2009
Winter solstice

Spring equinox

Summer solstice

Autumnal equinox

earth
Winter solstice

Spring equinox

Summer solstice

Autumnal equinox

89 days

89⅞ days

92⅓ days

93⅛ days
Winter solstice
89 days
Spring equinox
92¼ days
Summer solstice
Autumnal equinox
89⅞ days
93⅞ days
Basic problem of astronomy:

Given the arc of a circle, find the length of the chord that subtends this arc.
To measure an angle made by two line segments, draw a circle with center at their intersection.

The angle is measured by the distance along the arc of the circle from one line segment to the other.

The length of the arc can be represented by a fraction of the full circumference: $43^\circ = \frac{43}{360}$ of the full circumference.

If the radius of the circle is specified, the length of the arc can also be measured in the units in which the radius is measured:

Radius = 3438, Circumference = $60 \times 360 = 21600$

Radius = 1, Circumference = $2\pi$
Find the chord lengths for $3^\circ 42'$ and $52'$. The distance from the earth to the center of the sun’s orbit is found by taking half of each chord length and using the Pythagorean theorem.

Note that the chord lengths depend on the radius.
Ptolemy of Alexandria

*Circa* 85–165 CE

- Constructed table of chords in increments of $\frac{1}{2}^\circ$ and provided for linear interpolation in increments of $\frac{1}{2}$ minute.
- Values of chord accurate to 1 part in $60^3 = 216,000$ (approximately 7-digit accuracy).

Frontispiece from Ptolemy’s *Almagest*

Peurbach and Regiomantus edition of 1496
arc = 60°
chord = R
Euclid’s *Elements*
Book 13, Proposition 9

The ratio of the radius of a circle to the side of the regular inscribed decahedron is equal to the golden ratio ("mean and extreme proportion").

\[
\frac{R + x}{R} = \frac{R}{x}
\]

\[
\text{Crd } 36^\circ = \frac{\sqrt{5} - 1}{2} R
\]
Euclid’s *Elements*
Book 13, Proposition 10

The square of the side of the regular inscribed decahedron plus the square of the radius of the circle is equal to the square of the side of the regular inscribed pentagon.

\[
\frac{R}{y} = \frac{AC}{R} \implies R^2 = y \cdot AC
\]

\[
\frac{x}{BC} = \frac{y}{x} \implies x^2 = y \cdot BC
\]

\[
R^2 + x^2 = y^2
\]

\[
\text{Chord } 72^\circ = \sqrt{\frac{5 + \sqrt{5}}{2}} R
\]
Ptolemy’s Lemma: Given any quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.

\[ \overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC} \]
Ptolemy’s Lemma: Product of the diagonals equals the sum of the products of the opposite sides.

\[
AC = 2R \\
\angle ADC = \angle ABC = 90^0 \\
2R \cdot BD = AD \cdot BC + AB \cdot CD
\]
Ptolemy’s Lemma: Product of the diagonals equals the sum of the products of the opposite sides.

\[ AC = 2R \]

\[ \angle ADC = \angle ABC = 90^0 \]

\[ 2R \cdot BD = AD \cdot BC + AB \cdot CD \]

\[ 2R \cdot \text{Crd}\left(\alpha + \beta\right) \]

\[ = \text{Crd}\alpha \cdot \text{Crd}\left(180^0 - \beta\right) \]

\[ + \text{Crd}\beta \cdot \text{Crd}\left(180^0 - \alpha\right) \]
\[ \text{Crd } 60^0 \text{ and Crd } 72^0 \Rightarrow \text{Crd } 12^0 \]
If \( \alpha < \beta < 90^0 \), then
\[
\frac{\text{Crd } \beta}{\text{Crd } \alpha} < \frac{\beta}{\alpha}
\]
\[
\frac{1}{\beta} \text{Crd } \beta < \frac{1}{\alpha} \text{Crd } \alpha
\]
\[
\frac{2}{3} \text{Crd } 1^0 30' < \text{Crd } 1^0 < \frac{4}{3} \text{Crd } 45'
\]
\[
\Rightarrow \text{Crd } 1^0
\]
\[
\Rightarrow \text{Crd } 30'
\]
Kushan Empire

1st–3rd centuries CE

Arrived from Central Asia, a successor to the Seleucid Empire

Imported Greek astronomical texts and translated them into Sanskrit
**Surya-Siddhanta**

*Circa 300 CE*

Earliest known Indian work in trigonometry, had already made change from chords to half-chords

*Ardha-jya* = half bowstring

Became *jya* or *jiva*
Chord $\theta = \text{Crd } \theta = 2 \sin \frac{\theta}{2} = 2R \sin \frac{\theta}{2}$

Jiya – Sanskrit
Jiba (jyb) – Arabic
  jyb $\rightarrow$ jaib, fold or bay
Sinus, fold or hollow – Latin
Sine – English
$kotijya = \cosine$
Al-Khwarizmi, Baghdad
*Circa 790–840*

Earliest known reference to shadow length as a *function* of the sun’s elevation.
Al-Khwarizmi, Baghdad
*Circa* 790–840

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Earliest known reference to shadow length as a *function* of the sun’s elevation
Al-Khwarizmi, Baghdad  
*Circa 790–840*

Earliest known reference to shadow length as a *function* of the sun’s elevation
If you know the angle, θ, and the leg adjacent to that angle, the Tangent gives the length of the opposite leg.
Abu’l Wafa, Baghdad, 940–998

If you know the angle, \( \theta \), and the leg adjacent to that angle, the Tangent gives the length of the opposite leg.

The Cotangent is the Tangent of the complementary angle. If you know the angle and the length of the leg opposite that angle, the Cotangent gives the length of the adjacent leg.
Abu’l Wafa, Baghdad, 940–998

If you know the angle, \( \theta \), and the leg adjacent to that angle, the Tangent gives the length of the opposite leg.

The Cotangent is the Tangent of the complementary angle. If you know the angle and the length of the leg opposite that angle, the Cotangent gives the length of the adjacent leg.

\[
\text{Tangent} = \frac{\sin \theta}{\cos \theta} = \frac{R}{R} = \frac{\sin \theta}{\cos \theta}
\]

\[
\text{Cotangent} = \frac{\cos \theta}{\sin \theta} = \frac{R}{R} = \frac{\cos \theta}{\sin \theta}
\]
Abu’l Wafa, Baghdad, 940–998

If you know the angle, $\theta$, and the leg adjacent to that angle, the Secant gives the length of the hypotenuse.

$$\frac{\text{Sec } \theta}{R} = \frac{R}{\cos \theta}$$
Abu’l Wafa, Baghdad, 940–998

If you know the angle, $\theta$, and the leg adjacent to that angle, the Secant gives the length of the hypotenuse.

The Cosecant is the Secant of the complementary angle. If you know the angle and the length of the leg opposite that angle, the Cosecant gives the length of the hypotenuse.

$$\frac{\text{Csc} \, \theta}{R} = \frac{R}{\sin \theta}$$
Bartholomeo Pitiscus, 1561–1613, Grunberg in Silesia

1595 published *Trigonometria*, coining the term “trigonometry.”

According to Victor Katz, this was the first “text explicitly involving the solving of a real plane triangle on earth.”
Leonhard Euler, 1707–1783

Standardizes the radius of the circle that defines the angle to $R = 1$.

If we want to apply the tools of calculus, we need to measure arc length and line length in the same units, thus the circumference of the full circle is $2\pi$.

Euler did not use radians. For him, trigonometric functions expressed the lengths of lines in terms of the length of an arc of a circle of radius 1.
1840–1890

During this half-century, trigonometry textbooks shift from trigonometric functions as lengths of lines determined by arc lengths to ratios of sides of right triangles determined by angles.

For the first time, this necessitates a name for the angle unit used in calculus: radian. Coined independently in the 1870s by Thomas Muir and William Thomson (Lord Kelvin).
Lessons:

1. There is a lot of good and interesting circle geometry that sits behind trigonometry.
Lessons:

2. Ratios are intrinsically hard. It is much easier and more intuitive to think of trigonometric functions as lengths of lines.

3. Angle as a measurement of how far something has turned is intrinsically hard. It is easier to think of trigonometric functions as functions of arc length.

4. Rather than trying to deal with radians as the numerator in a fraction whose denominator is $2\pi$, think of them as the distance traveled along the circumference of a circle of radius 1.
Lessons:

5. The circle definition of the sine and cosine is much closer to the way these functions have been defined and used throughout history than is soh-cah-toa.

\[ R = 1 \]

\[ \sin \theta \]

\[ \cos \theta \]

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