CALCULUS CLASSROOM CAPSULES

SESSION S186

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Calculus Classroom Capsules
Session S186

Overview

I. Finite Differences of Polynomials and Exponential Functions

II. Logarithmic Differentiation: Two Wrongs Make a Right

III. Product and Quotient Rules for Derivatives: Using Logarithmic Differentiation

IV. Critical Points of Polynomial Functions

V. An Overlooked Calculus Question

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I. Finite Differences of Polynomials and Exponential Functions

Before you begin to teach the derivative of an exponential function consider a motivational exercise to compute finite differences of polynomials and exponential functions.

1. Consider the set of data $\{(x_i, y_i)\}_{i=1}^{10}$ given in the first two columns of Table (1)

The data $\{(x_i, y_i)\}_{i=1}^{10}$ is non-linear.

In fact it is quadratic: $u(x) = ax^2 + bx + c$

Its regression equation is $u(x) = 0.5\,x^2 + 0.5x$ with $R^2 = 1$

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Table (1)

x	y	1st Finite Differences	2nd Finite Differences	3rd Finite Differences
		$u_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$ $i = 1, \ldots, 9$	$v_i = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$ $i = 1, \ldots, 8$	$w_i = \frac{v_{i+1} - v_i}{x_{i+1} - x_i}$ $i = 1, \ldots, 7$
1	1	$\frac{3-1}{2-1} = 2$	$\frac{3-2}{2-1} = 1$	$\frac{1-1}{2-1} = 0$
2	3	$\frac{6-3}{3-2} = 3$	$\frac{4-3}{3-2} = 1$	$\frac{1-1}{3-2} = 0$
3	6	4	1	0
4	10	5	1	0
5	15	6	1	0

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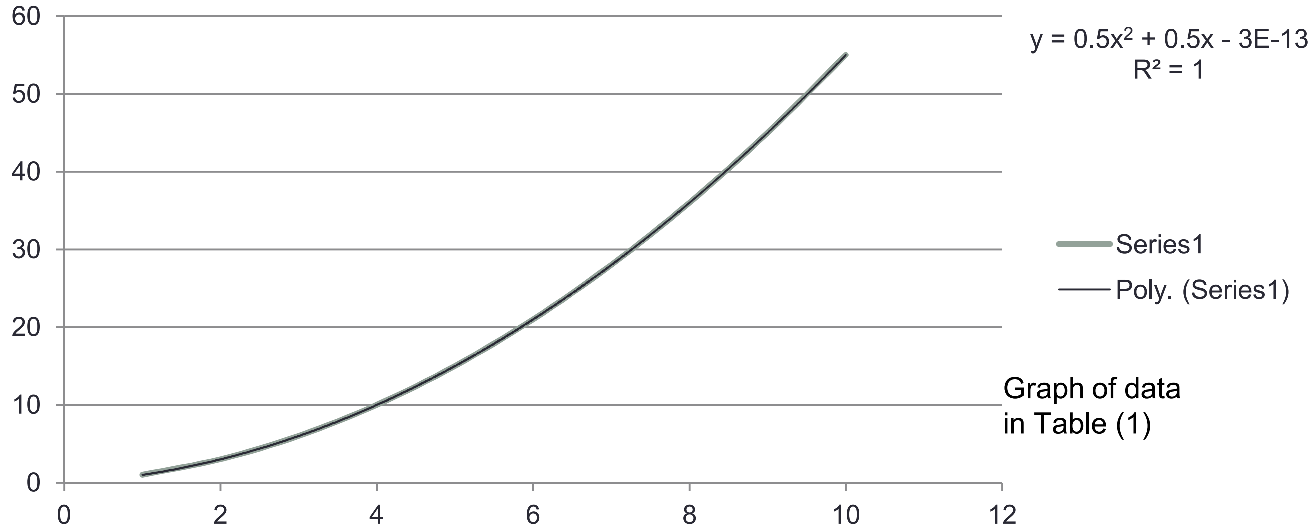

Graph of data in Table (1)

The column of first finite difference is linear:

$$\frac{(ax_{i+1}^2 bx_{i+1}+c)-(ax_i^2+bx_i+c)}{x_{i+1}-x_i}=\frac{a\left(x_{i+1}^2-x_i^2\right)+b(x_{i+1}-x_i)+c-c}{x_{i+1}-x_i}$$

$$=\frac{(x_{i+1}-x_i)(a(x_{i+1}+x_i)+b)}{x_{i+1}-x_i}=a(x_{i+1}+x_i)+b$$

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The column of second finite differences is constant:

$$\frac{mx_{i+1}+b-(mx_i+b)}{x_{i+1}-x_i}=\frac{m(x_{i+1}-x_i)+b-b}{x_{i+1}-x_i}=m$$

The column of third finite differences is 0:

$$\frac{m-m}{x_{i+1}-x_i}=0$$

Finite differences are difference quotients and are a good motivation for the derivative of polynomial functions and for analyzing data sets in general

2. Consider the second set of data $\{(x_i, y_i)\}_{i=1}^{10}$ in Table (2) below

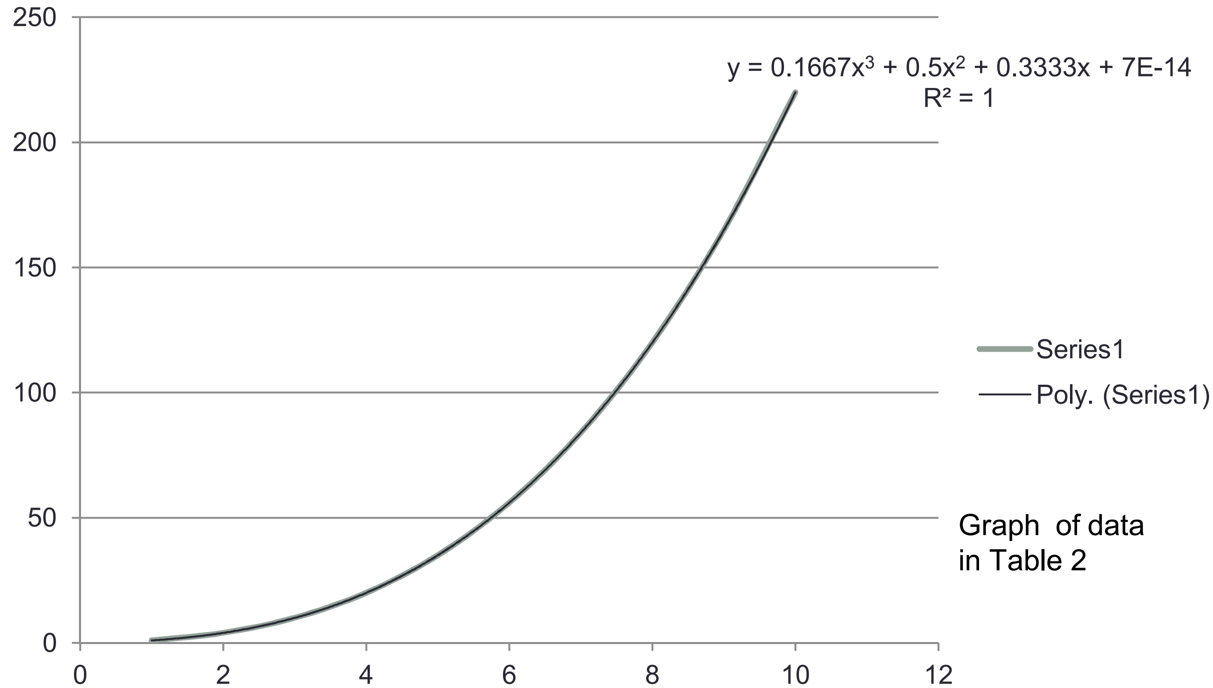
Table (2)

x	y	1st Finite Differences	2nd Finite Differences	3rd Finite Differences	4th Finite Differences
		$u_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$ $i = 1, \dots, 9$	$v_i = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$ $i = 1, \dots, 8$	$w_i = \frac{v_{i+1} - v_i}{x_{i+1} - x_i}$ $i = 1, \dots, 7$	$z_i = \frac{w_{i+1} - w_i}{x_{i+1} - x_i}$ $i = 1, \dots 6$
1	1	$\frac{4-1}{2-1} = 3$	$\frac{6-3}{2-1} = 3$	$\frac{4-3}{2-1} = 1$	$\frac{1-1}{2-1} = 0$
2	4	$\frac{10-4}{3-2} = 6$	$\frac{10-6}{3-2} = 4$	$\frac{5-4}{3-2} = 1$	$\frac{1-1}{3-2} = 0$
3	10	10	5	1	0
4	20	15	6	1	0
5	35	21	7	1	0

A regression on the first two columns will result in a cubic model

$$y = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x \quad with \;\; R^2 = 1$$

As in the first example we can show that the first finite difference will be quadratic, the second will be linear, the third will be constant and the fourth will be zero



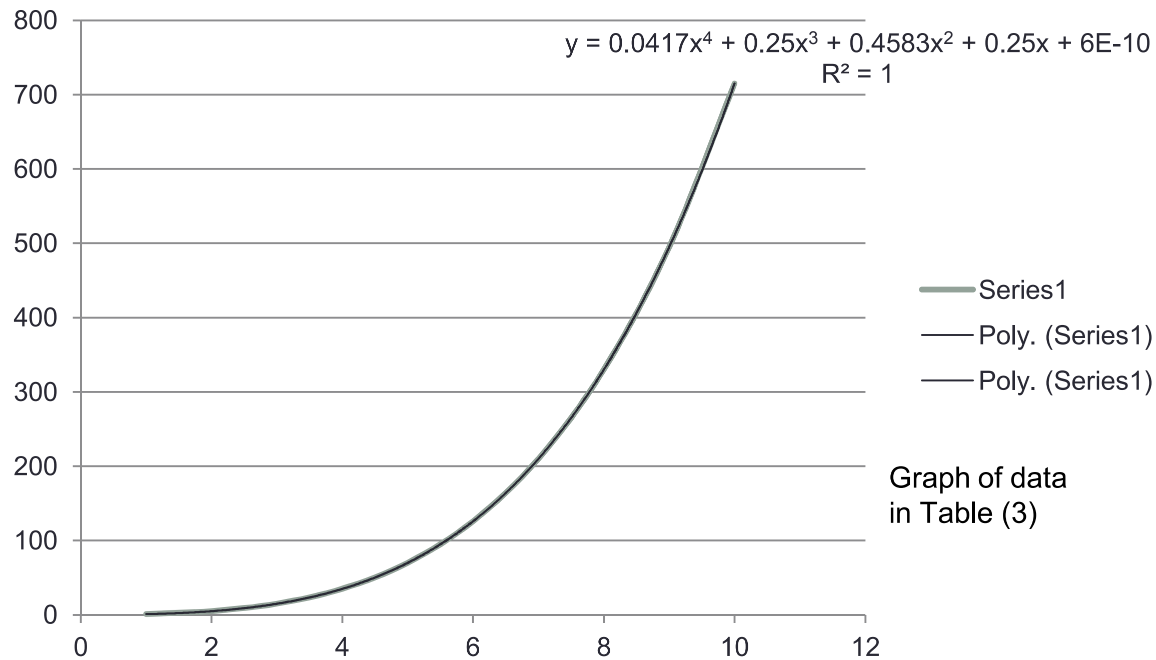
Graph of data in Table 2

3. Consider the third set of data $\{(x_i, y_i)\}$, i = 1…10 in Table (3) below

Table (3)

x	y	1st Finite Differences	2nd Finite Differences	3rd Fi4nite Differences	4th Finite Differences	5th Finite Differences
		$u_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$ $i = 1,\ldots,9$	$v_i = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$ $i = 1,\ldots,8$	$w_i = \frac{v_{i+1} - v_i}{x_{i+1} - x_i}$ $i = 1,\ldots,7$	$z_i = \frac{w_{i+1} - w_i}{x_{i+1} - x_i}$ $i = 1,\ldots,6$	$t_i = \frac{z_{i+1} - z_i}{x_{i+1} - x_i}$ $i = 1,\ldots,5$
1	1	$\frac{5-1}{2-1} = 4$	$\frac{10-4}{2-1} = 6$	$\frac{10-6}{2-1} = 4$	$\frac{5-4}{2-1} = 1$	$\frac{1-1}{2-1} = 0$
2	5	$\frac{15-5}{3-2} = 10$	$\frac{20-10}{3-2} = 10$	$\frac{15-10}{3-2} = 5$	$\frac{6-5}{3-2} = 1$	$\frac{1-1}{3-2} = 0$
3	15	20	15	6	1	0
4	35	35	21	7	1	0
5	70	56	28	8	1	0
6	126	84	36	9	1	0

The regression equation on the first two columns results in a fourth degree equation shown below. $y = \frac{1}{24}x^4 + \frac{1}{4}x^3 + \frac{11}{24}x^2 + \frac{1}{4}x$



Graph of data in Table (3)

A further analysis, as was done in the first example, will show that the column of first finite differences represents cubic data; the second quadratic data; the third linear data; the fourth constant data and the and the fifth represents the 0 function.

The three examples above motivate average rate of change and the derivative of polynomials.

Now consider the exponential data in Table (4)

Table (4)

x	y	1st Finite Differences	2nd Finite Differences	3rd Finite Differences	4th Finite Differences	5th Finite Differences
		$u_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$ $i = 1, \ldots, 9$	$v_i = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$ $i = 1, \ldots, 8$	$w_i = \frac{v_{i+1} - v_i}{x_{i+1} - x_i}$ $i = 1, \ldots, 7$	$z_i = \frac{w_{i+1} - w_i}{x_{i+1} - x_i}$ $i = 1, \ldots, 6$	$t_i = \frac{z_{i+1} - z_i}{x_{i+1} - x_i}$ $i = 1, \ldots, 5$
1	2	$\frac{4-2}{2-1} = 2$	$\frac{4-2}{2-1} = 2$	$\frac{4-2}{2-1} = 2$	$\frac{4-2}{2-1} = 2$	$\frac{4-2}{2-1} = 2$
2	4	$\frac{8-4}{3-2} = 4$	$\frac{8-4}{3-2} = 4$	$\frac{8-4}{3-2} = 4$	$\frac{8-4}{3-2} = 4$	$\frac{8-4}{3-2} = 4$
3	8	8	8	8	8	8
4	16	16	16	16	16	16
5	32	32	32	32	32	32
6	64	64	64	64	64	64

Unlike polynomial functions, finite differences of exponential functions remain exponential

$$\frac{2^{i+1} - 2^i}{2-1} = \frac{2^i(2-1)}{1} = 2^i$$

Consider the following exponential data of powers of 3 in Table (5)

Table (5)

x	y	1st Finite Differences	2nd Finite Differences	3rd Finite Differences	4th Finite Differences	5th Finite Differences
		$u_i=\frac{y_{i+1}-y_i}{x_{i+1}-x_i}$ $i=1,\dots,9$	$v_i=\frac{u_{i+1}-u_i}{x_{i+1}-x_i}$ $i=1,\dots,8$	$w_i=\frac{v_{i+1}-v_i}{x_{i+1}-x_i}$ $i=1,\dots,7$	$z_i=\frac{w_{i+1}-w_i}{x_{i+1}-x_i}$ $i=1,\dots,6$	$t_i=\frac{z_{i+1}-z_i}{x_{i+1}-x_i}$ $i=1,\dots,5$
1	3	$\frac{9-3}{2-1}=2\cdot3$	$\frac{2\cdot3^2-2\cdot3}{2-1}=2^2\cdot3$	$\frac{2^4\cdot3^2-2^4\cdot3}{2-1}=2^5\cdot3$	$\frac{2^3\cdot3^2-2^3\cdot3}{2-1}=2^4\cdot3$	$\frac{2^4\cdot3^2-2^4\cdot3}{2-1}=2^5\cdot3$
2	9	$2\cdot3^2$	$2^2\cdot3^2$	$2^3\cdot3^2$	$2^4\cdot3^2$	$2^5\cdot3^2$
3	27	$2\cdot3^3$	$2^2\cdot3^3$	$2^3\cdot3^3$	$2^4\cdot3^3$	$2^5\cdot3^3$
4	81	$2\cdot3^4$	$2^2\cdot3^4$	$2^3\cdot3^4$	$2^4\cdot3^4$	$2^5\cdot3^4$
5	243	$2^5\cdot3^5$	$2^2\cdot3^5$	$2^3\cdot3^3$	$2^4\cdot3^5$	$2^5\cdot3^5$
6	729	$2\cdot3^6$	$2^2\cdot3^6$	$2^3\cdot3^6$	$2^4\cdot3^6$	$2^5\cdot3^6$

- Exponential functions remain exponential when computing their finite differences. In this case, a constant times an exponential function remains an exponential function.

- Compare with the derivative of an exponential function y = e^x which after this exercise comes with no surprise for the student as y' = e^x

- Compare outcome of Table (5) with repeated derivative of $y = a^x$:
  $y' = a^x \ln a$; $y'' = a^x (\ln a)^2$; $y^{(n)} = a^x (\ln a)^n$ In both cases the exponential function is multiplied by a constant.

- Finite differences are a good motivation for the derivative of exponential functions.

II. Logarithmic Differentiation: Two Wrongs Make a Right

Brannen and Ford write about their experience in Calculus I class teaching the derivative of $y = x^x$ and contribute an important insight from that experience.

Some students view $y = x^x$ as $y = x^n$, a power function and differentiate it as $y' = xx^{x-1}$ (Wrong!)

Other students view it as an exponential function such as $y = a^x$ since the exponent is a variable and differentiate it as $y' = x^x \ln x$ (Also wrong!)

So $y = x^x$ is neither a power function nor an exponential function.

As teachers we yearn for the student who will write it as $y = e^{\ln x^x}$

At this time, Brannon and Ford are about to introduce logarithmic differentiation when a student raises her hand and suggests that

$y = x^x$ is both a power function and an exponential function so its derivative must be the sum of the two namely,

$$y' = x x^{x-1} + x^x \ln x \qquad (1)$$

At this point they use the "correct method" logarithmic differentiation to get

$$\ln y = \ln x^x = x \ln x$$

$$\frac{y'}{y} = (1)\ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

$$y' = y(\ln x + 1) = x^x(\ln x + 1) \qquad (2)$$

But equation (1) and equation (2) turned up to be exactly the same. That was not the expected outcome!!!!. They assured the class that that was just a coincidence!

Using the student's suggested method of the derivative of y = $f(x)^{g(x)}$ in general, they got:

$$y' = g(x)f(x)^{g(x)-1}f'(x) - f(x)^{g(x)}g'(x)\ln(f(x))$$

Which can also be written as:

$$y' = g(x)f(x)^{g(x)}\frac{f'(x)}{f(x)} - f(x)^{g(x)}g'(x)\ln(f(x)) \qquad (3)$$

Using logarithmic differentiation they got

$$\ln y = \ln f((x)^{g(x)}) = g(x)\ln(f(x))$$

$$\frac{y'}{y} = g(x)\frac{f'(x)}{f(x)} - g'(x)\ln(f(x))$$

$$y' = f(x)^{g(x)}\left(g(x)\frac{f'(x)}{f(x)} - g'(x)\ln(f(x))\right)$$

Which when the y = $f(x)^{g(x)}$ is distributed is exactly the same as the student's method

$$y' = g(x)f(x)^{g(x)}\frac{f'(x)}{f(x)} - f(x)^{g(x)}g'(x)\ln(f(x)) \qquad (4)$$

So it was not a coincidence. Two wrongs indeed make a right but why????

Why can one treat g(x) as a constant and then treat f(x) as a constant?

This is precisely what one does with partial differentiation namely, holding one variable constant while differentiating the other. $y = f(x)^{g(x)}$ is a function of two variables (f, g) each of which is function of x. So using the multivariable chain rule we get

$$\frac{dy}{dx} = \frac{\delta y}{\delta f}\frac{df}{dx} + \frac{\delta y}{\delta g}\frac{dg}{dx}$$

$$\frac{dy}{dx} = g(x)f(x)^{g(x)-1}f'(x) + f(x)^{g(x)}g'(x)\ln(f(x)) \qquad (5)$$

Equation (5) is the same as (3) and (4). The student is vindicated.

But continue to teach logarithmic differentiation.

III. Alternative Proof of the Product and the Quotient Rule for Derivatives

Most calculus books use the formal definition of the derivative to prove the product and quotient rules. In some cases the product and quotient rules can be proved using logarithmic differentiation

The Product Rule

If $f(x) = u(x)v(x)$ such that $u(x) > 0$ *and* $v(x) > 0$ for all x in the domain of f, and u and v are differentiable functions of x, then

$\ln f(x) = \ln[u(x)v(x)] = \ln u(x) + \ln v(x)$ Taking the derivative of both sides yields:

$$\frac{f'(x)}{f(x)} = \frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)}$$

$$f'(x) = f(x)\left[\frac{v(x)u'(x) + u(x)v'(x)}{u(x)v(x)}\right] = u(x)v(x)\left[\frac{v(x)u'(x) + u(x)v'(x)}{u(x)v(x)}\right]$$

$$f'(x) = v(x)u'(x) + u(x)v'(x).$$

The Quotient Rule

$If\ f(x) = \frac{u(x)}{v(x)}\ where\ u(x)\ and\ v(x)$ are defined as above,

$Then\ \ln f(x) = \ln \frac{u(x)}{v(x)} = \ln u(x) - \ln v(x)$ Differentiating both sides,

$\frac{f'(x)}{f(x)} = \frac{u'(x)}{u(x)} - \frac{v'(x)}{v(x)} = \frac{v(x)u'(x) - u(x)v'(x)}{u(x)v(x)}$ Multiply both sides by $f(x)$ to get

$$f'(x) = f(x)\left[\frac{v(x)u'(x) - u(x)v'(x)}{u(x)v(x)}\right] = \frac{u(x)}{v(x)}\left[\frac{v(x)u'(x) - u(x)v'(x)}{u(x)v(x)}\right]$$

$$f'(x) = \left[\frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}\right]$$

Finding the derivative of a product and quotient functions under the conditions stated gives an alternative useful method for reinforcing the product and quotient rules for derivatives and provides good practice and incentive for using logarithmic differentiation.

IV. Critical Points of Polynomial Functions[4]

Nowadays we use computers and graphing calculators more often to provide opportunities for experimentation. In this activity, suitable for a student in a calculus I class, critical points of a one parameter family of polynomials will be investigated and a general proof of how to obtain the locus of the $\{f_t(x)\}$ critical points for each family will be shown.

Example 1. Find the critical points of the family of polynomials

$$f_t(x) = tx^4 - 6x^2 + 4$$

If $(x, f_t(x))$ is a critical point of f_t then $f_t'(x) = 0$ implies that $4tx^3 - 12x = 0$

Then $x = 0 \ or \ t = \frac{3}{x^2}$ In addition $4tx^3 = 12x$ implies that $4tx^4 = 12x^2$

Therefore $f_t(x) = tx^4 - 6x^2 + 4 = \frac{1}{4}(4tx^4 - 24x^2 + 16) = \frac{1}{4}(12x^2 - 24x^2 + 16) = \frac{1}{4}(-12x^2 + 16)$

This shows that every critical point $(x, f_t(x))$

lies on the graph of the polynomial $p(x) = \frac{1}{4}(-12x^2 + 16)$

Conversely,

If $(x, p(x))$ is an arbitrary point on the graph of $p(x)=\frac{1}{4}(-12x^2+16)$

and if x = 0 then (x, p(x)) = (0, 4) which is also a critical point of f_t for every t.

If $x \neq 0$ then (x, p(x)) is a critical point of f_t when $t = \frac{3}{x^2}$ as derived above.

Therefore the locus of critical points of the family $(x, f_t(x))$ is given by the graph of $p(x)=\frac{1}{4}(-12x^2-16)$

Figure (1) shows the graph of the function $f_t(x) = tx^4 - 6x^2 + 4 \;\; for \;\; t = 1, 2, 3, \ldots 10$ and the locus of critical points $p(x) = \frac{1}{4}(-12x^2 - 16)$

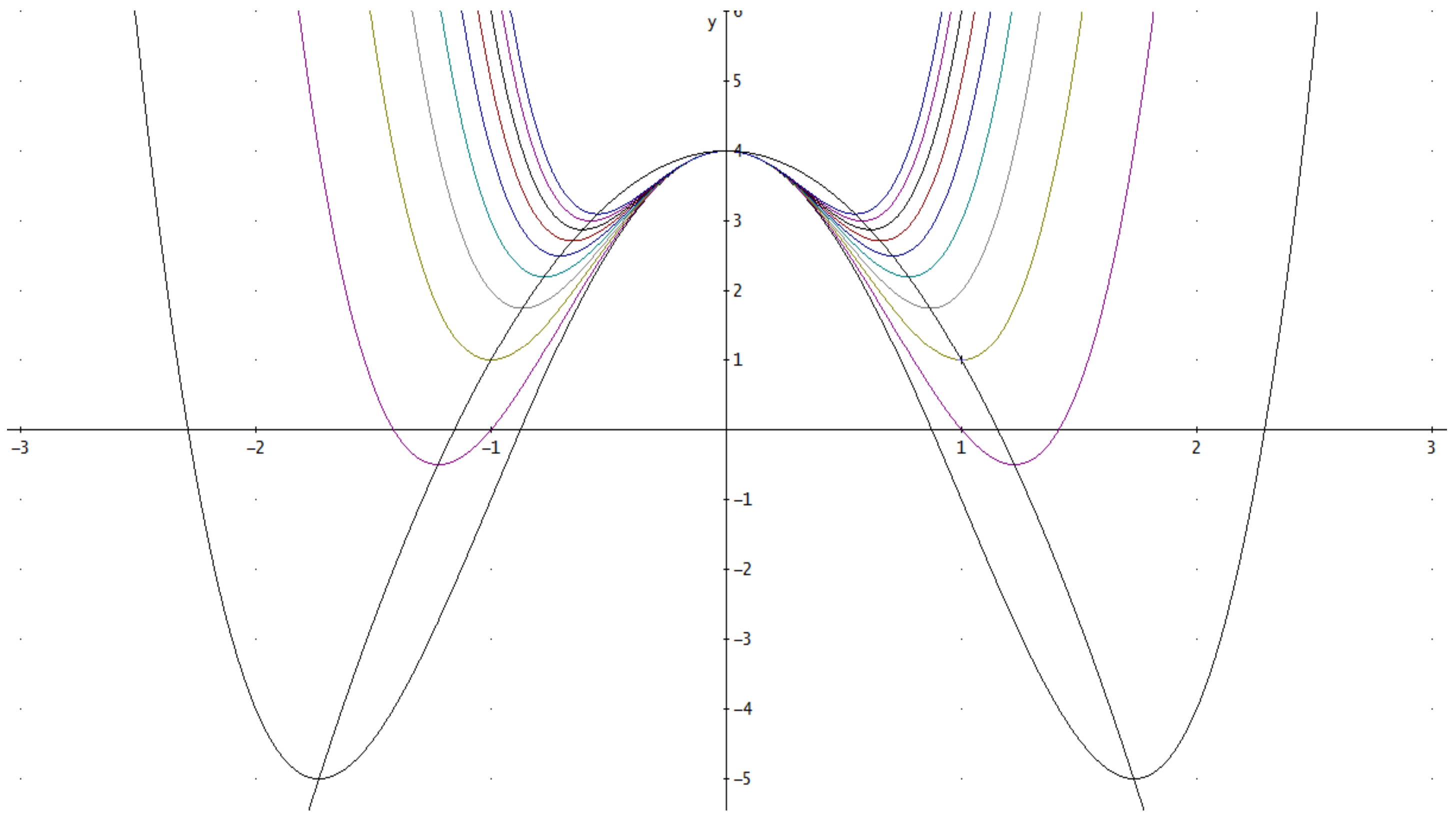


Figure for Example 1

This result can be generalized.

Theorem:

If $f_t(x)$ is a one parameter family of polynomials

of the form $f_t(x) = g(x) + tx^m$ where $m \geq 1$ and g(x) is a nonzero polynomial with no mth degree term, then the locus of critical points of $(x, f_t(x))$ is the graph of the polynomial $p(x)$ together with the point (0, g(0)) in case g(x) has no first degree term where

$$p(x) = g(x) - \frac{g'(x)x}{m} \quad for \;\; x \neq 0$$

Proof:

Suppose m = 1. If $(x, f_t(x))$ is a critical point of $f_t(x) = g(x) + tx$

then $f_t'(t) = g'(x) - t = 0$ and $t = -g'(x) \ \text{and} \ tx = -g'(x)x$

Therefore if $(x, f_t(x))$ is a critical point of f_t then $f_t(x) = g(x) - tx = g(x) - g'(x)x$ as required to show for m = 1.

Conversely, for every x, the point $(x, g(x) - g'(x)x)$ is a critical point of f_t when $t = -g'(x)$

Therefore the locus of critical points is given by the equation

$p(x) = g(x) - \frac{g'(x)x}{m} \ \text{for} \ x \neq 0$ for m = 1

Suppose $m \geq 2$. If $(x, f_t(x))$ is a critical point of $f_t(x) = g(x) - tx^m$

Then $f_t'(x) = g'(x) - tmx^{m-1} = 0$ and $tmx^{m-1} = -g'(x)$

Multiply both sides of last equation by x to get

$$tmx^m = -g'(x)x \;\; and \;\; tx^m = \frac{-g'(x)x}{m}$$

Therefore, if $(x, f_t(x))$ is a critical point of f_t then

$$f_t(x) = g(x) + tx^m = g(x) - \frac{g'(x)x}{m}$$

Conversely, consider an arbitrary point $\left(x, g(x) - \dfrac{g'(x)x}{m}\right)$

$g'(0) = 0$

$\left(x, g(x) - \dfrac{g'(x)x}{m}\right) = (0, g(0))$

f_t

Three cases should be observed:

1. If x = 0 and g has no first degree term then $g'(0)=0$

   and $\left(x, g(x)-\frac{g'(x)x}{m}\right)=(0, g(0))$ is a critical point of f_t for all t

2. If x = 0 and g has a first degree term then $g'(0)\neq 0$ and (0, g(0)) is not a critical point of any f_t

3. If $x\neq 0$ then $\left(x, g(x)-\frac{g'(x)x}{m}\right)$ is a critical point of f_t when $t=-\frac{g'(x)}{mx^{m-1}}$

This completes the proof.

From the first example and the theorem $f_t(x) = g(x) + tx^m$ m and g(x) are given; it is required to find $p(x) = g(x) - \frac{1}{m} g'(x)x$

Note that: First, if m=0 in the theorem $f_t(x) - g(x) - t$ then g(x) is a polynomial with no constant term. So the graph of $f_t(x)$ is a vertical translation of the graph of g(x). The locus of critical points $(x, f_t(x))$ of the family $\{f_t\}$ consists of vertical lines passing through the critical points of g.

Second, if the degree of $f_t(x)$ is n, then the locus of critical points p(x) is a polynomial

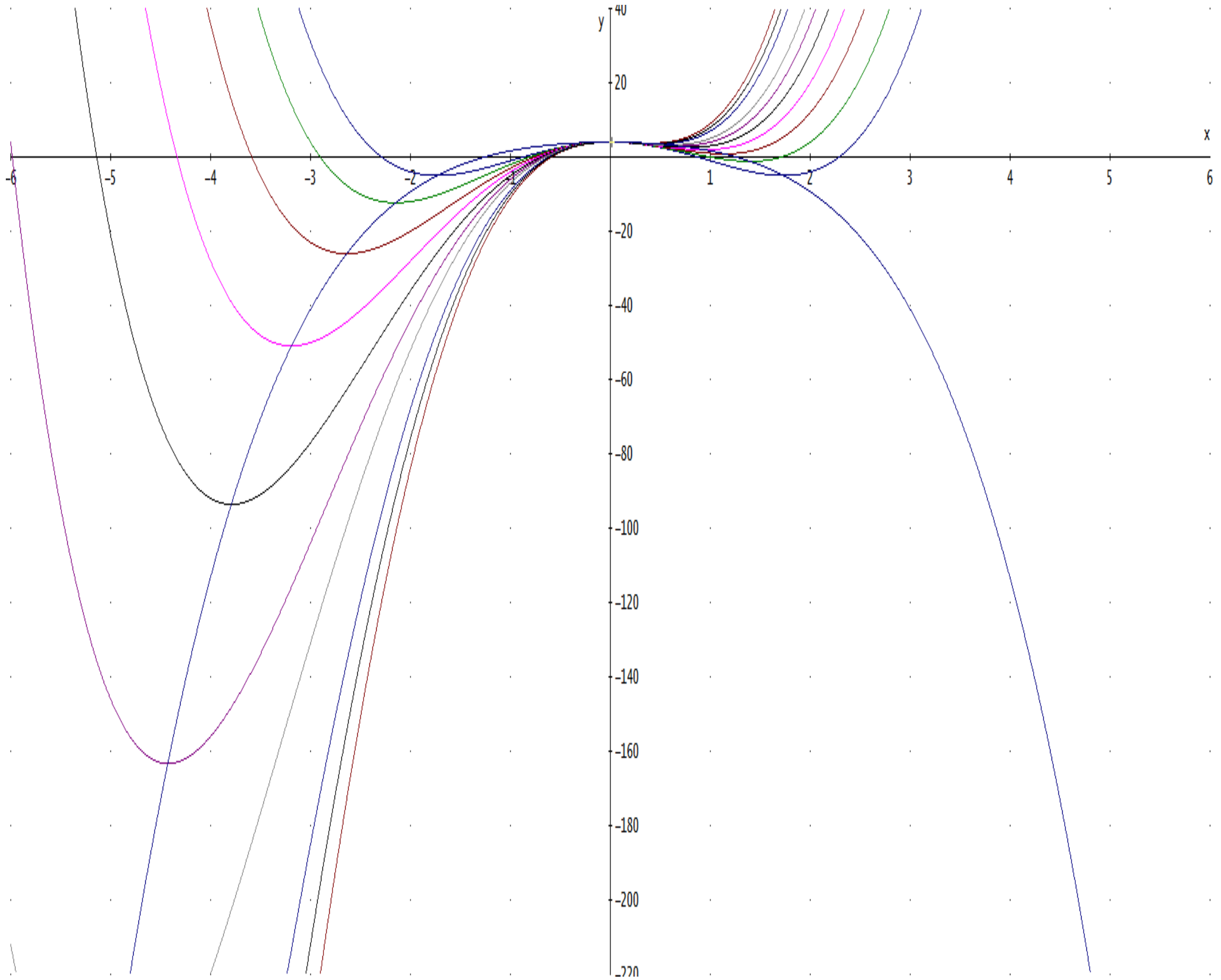
with $\text{degree}(p(x)) = n = \text{degree}(g(x))\ if\ m \neq n,$

and $\text{degree}(p(x)) = \text{degree}(g(x)) < n\ if\ m = n$

Example 2

Given			Required to Find
$f_t(x) = g(x) - tx^m$	m=?	$g(x)$	$p(x) = g(x) - \frac{1}{m} g'(x)x$
$f_t(x) = x^4 - tx^3 - 6x^2 - 4$ $t = 0, 1, 2, \ldots, 10$	m=3	$g(x) = x^4 - 6x^2 + 4$	$p(x) = \frac{1}{3}(-x^4 - 6x^2 + 12)$

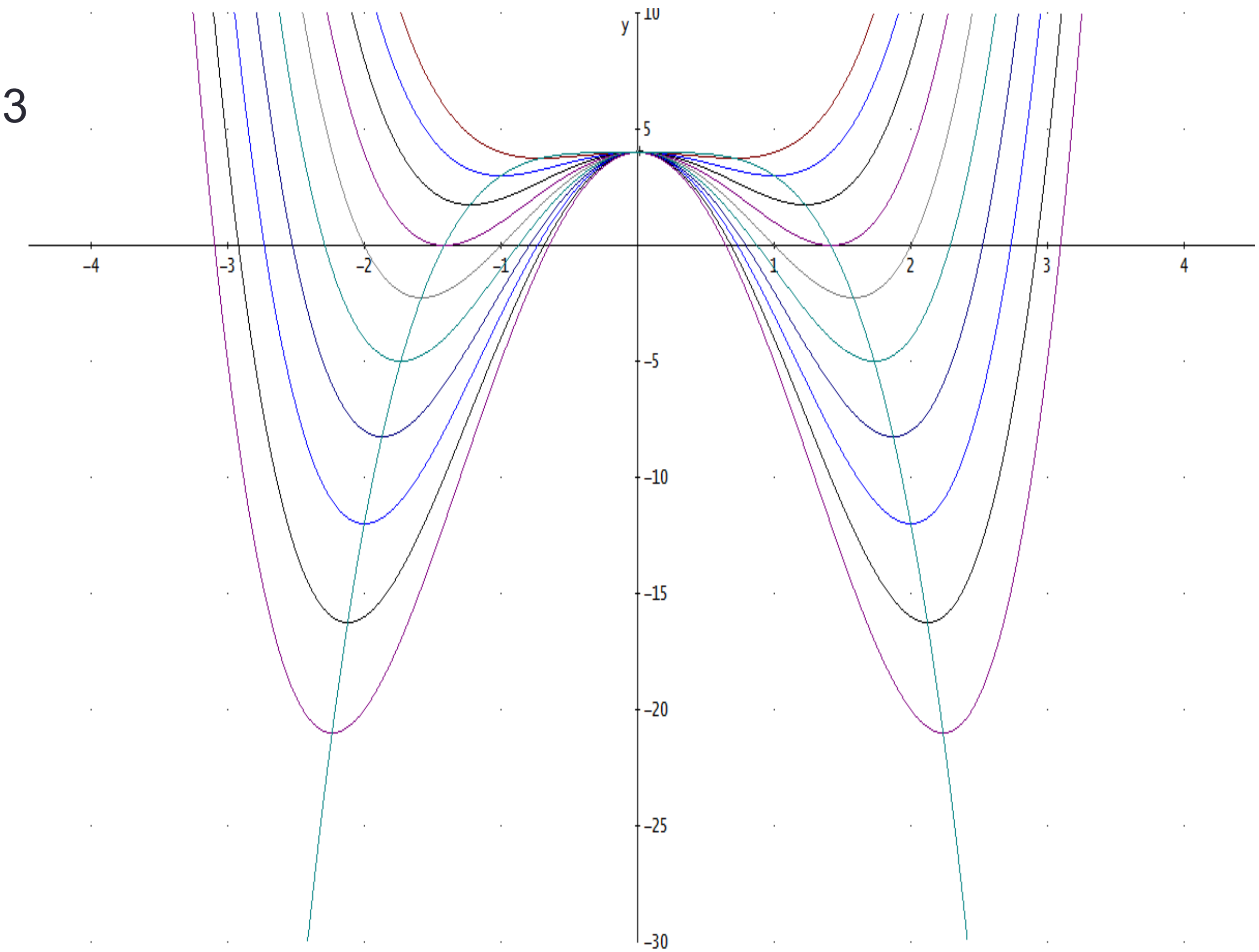
Figure for Example 2



Example 3

Given			Required to Find
$f_t(x)=g(x)+tx^m$	m=?	$g(x)$	$p(x)=g(x)-\frac{1}{m}g'(x)x$
$f_t(x)=x^4+tx^2+4$ $t=-1,-2,\ldots,-10$	m=2	$g(x)=x^4-4$	$p(x)=\frac{1}{2}(-2x^4+8)$

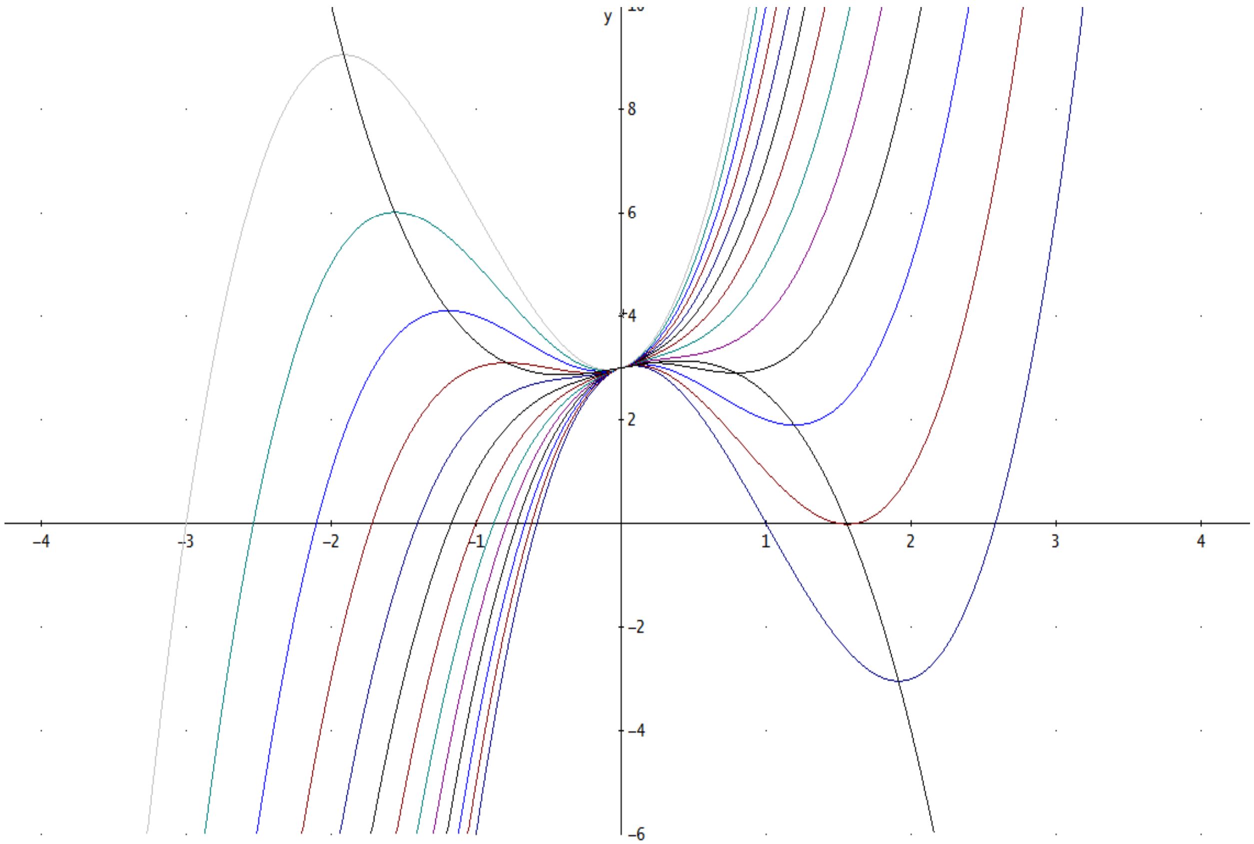
Figure for Example 3



Example 4

Given			Required to Find
$f_t(x)=g(x)+tx^m$	m=?	$g(x)$	$p(x)=g(x)-\frac{1}{m}g'(x)x$
$f_t(x)=2x^3+tx^2+x+3$ $t=-6,-5,-4,\ldots,4,5,6$	m=2	$g(x)=2x^3+x+3$	$p(x)=\frac{1}{2}(-2x^3-x+6)$ $x\neq 0$

Figure for Example 4



V. An Overlooked Calculus Question

Eugene Couch of the University of Calgary, in Alberta, Canada, notes that when introducing and teaching exponential functions of the form $f(x) = a^x$ and their inverse functions $g(x) = \log_a x$ the graphs that are drawn in most Precalculus and Calculus texts give the impression that these two functions do not intersect. Let us then consider the two questions:

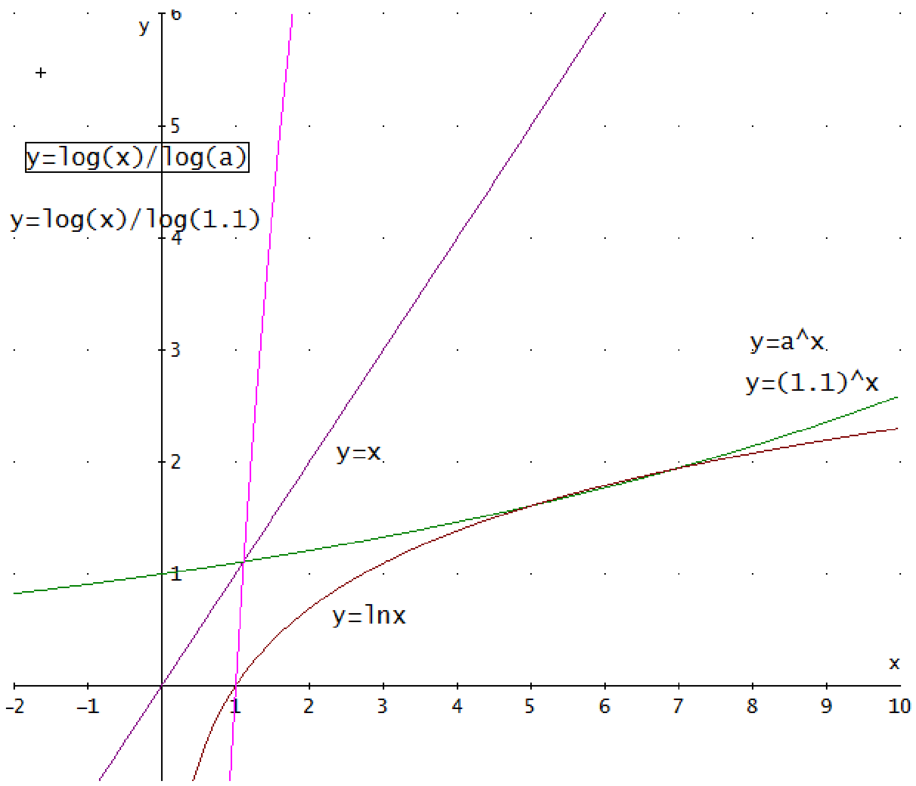
1) Do the graphs of $f(x) = a^x$ and $g(x) = \log_a x$ for $a > 1$, ever intersect?
2) If they do intersect, for what bases a and for what values of x does this happen?

Clearly they must intersect for some value of $a > 1$. If a is sufficiently close to 1, such as $a = 1.1$, the graph of $f(x) = a^x$ stays close to 1 for x as large as we like. So it must be intersected by $y = \ln x$ and also by $g(x) = \log_a x$ because $g(x) = \log_a x$ lies above $y = \ln x$ for $x > 1$ when $a < e$.

This graph shows $f(x) = a^x = 1.1^x$ intersecting $g(x) = \log_{1.1} x$ at $x = 1$.
In addition, it shows $y = \ln x$ intersecting $f(x) = a^x = 1.1^x$ and $g(x) = \log_{1.1} x$.

Note that $g(x) = \log_{1.1} x$ is above $y = \ln x$ for $a = 1.1 < e$.

Figure 5: $a = 1.1$, $a < e$



Observe algebraically that

$a^x > x$ implies $a^x > x > \log_a x$, and

$a^x < x$ implies $a^x < x < \log_a x$

Furthermore,

if $a^x = x$ then $\log_a x = x$ and

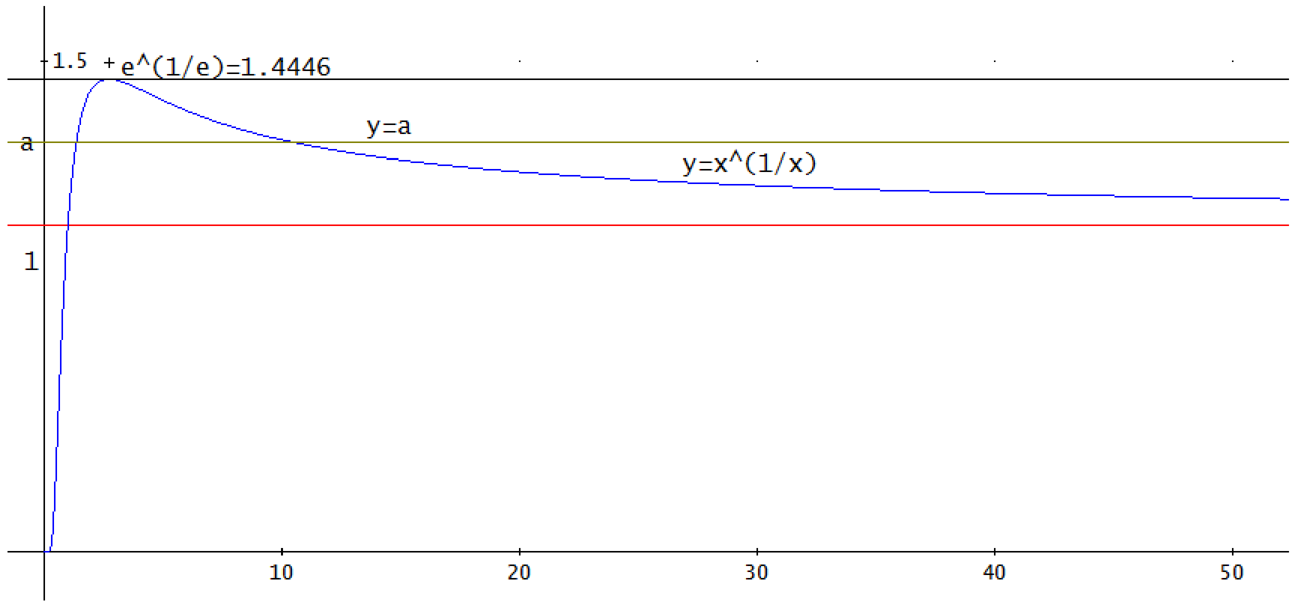
the graphs of $f(x) = a^x$ and $g(x) = \log_a x$ intersect at x

Thus the graphs of $f(x) = a^x$ and $g(x) = \log_a x$ intersect at x if and only if $a^x = x$

which is equivalent to $a = x^{1/x}$

The three graphs shown below are: $y = x^{1/x}$, $y = a = 1.25$ and $y = 1$ where $1 < a < e^{1/e} = 1.4446\ldots$

Figure 6



From the graph, it is evident that for $1 < a < e^{1/e} = 1.4446\ldots$

$f(x) = a^x$ and $g(x) = \log_a x$ intersect exactly twice

and exactly once at $x = e$ for $a = e^{1/e}$

The above analysis gives rise to the following three cases for a

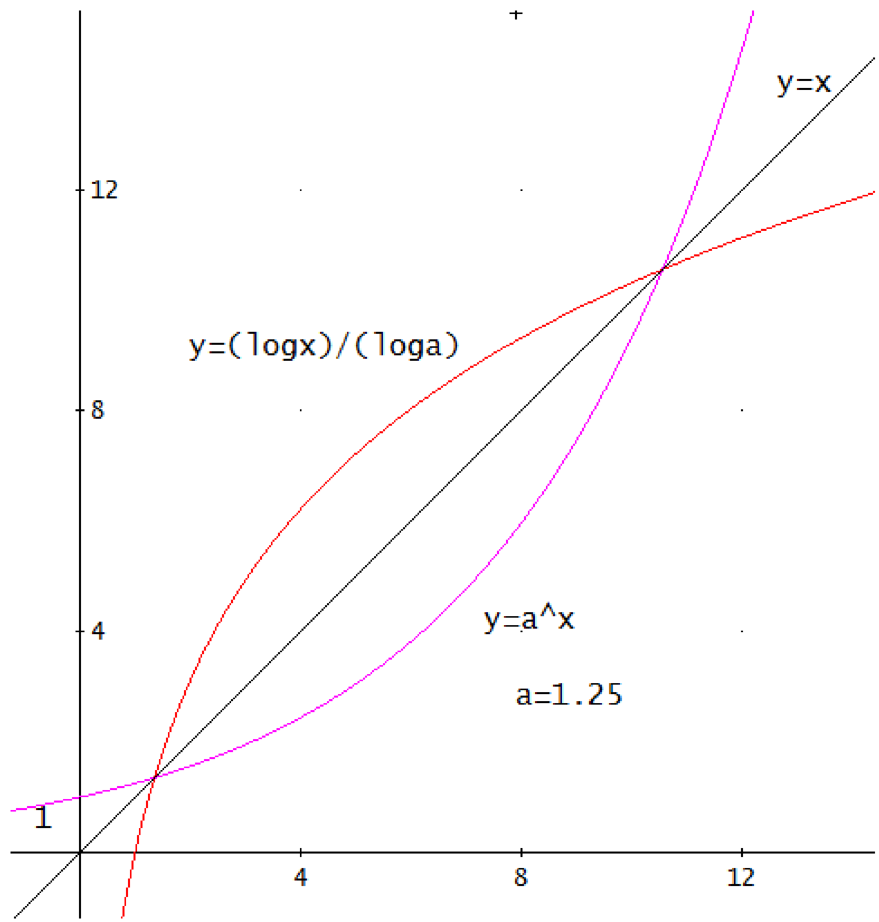
Case 1: $1 < a < e^{1/e}$
there are two Intersection points.

Figure 7: Graphs of $y = a^x$
$y = \log_a x$ and $y = x$ for $a = 1.25$

Intersection points are:

(x=1.3522, y=1.3522) and

(x = 10.56527, y =10.56527)



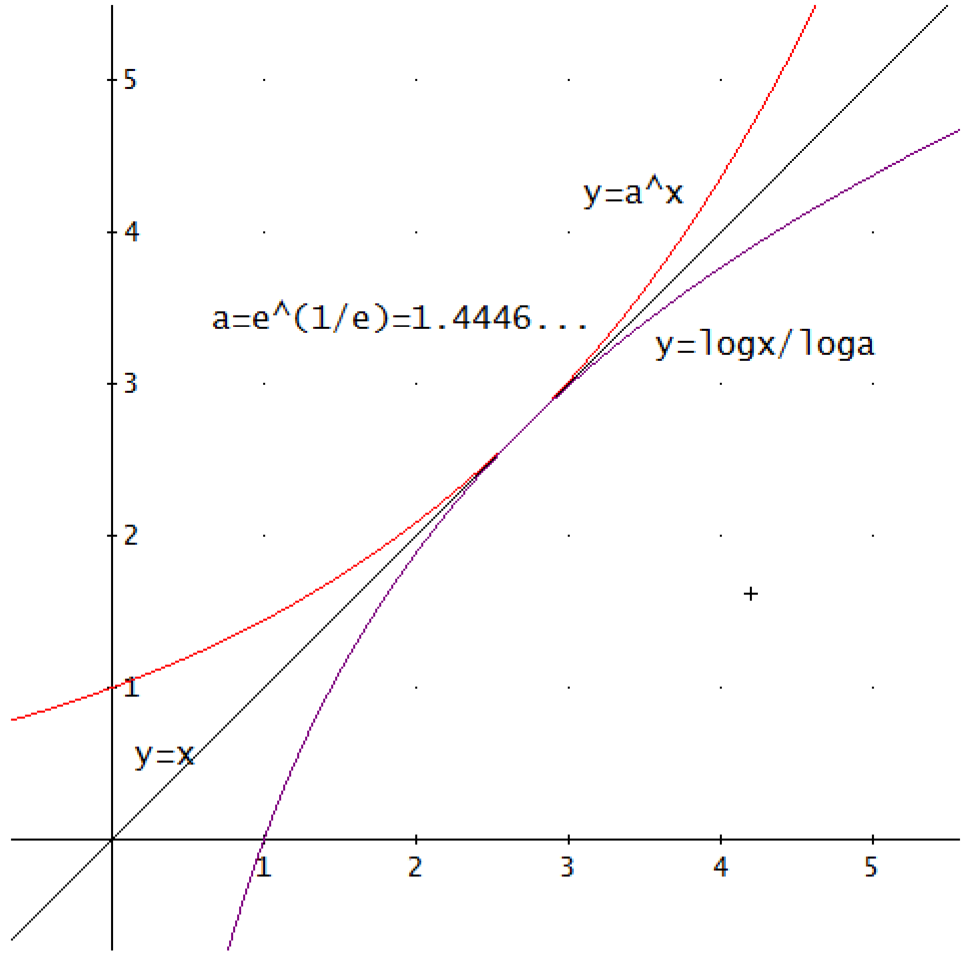
Case 2. For a = $e^{1/e}$ = 1.4446… there is one Intersection point.

Figure 8: Graphs of $y = a^x$

$y = \log_a x$ and $y = x$ for a = 1.4446…

Intersection point is at (x = e, y = e)

Note that in cases 1 and 2 all intersection points lie on the y = x line.

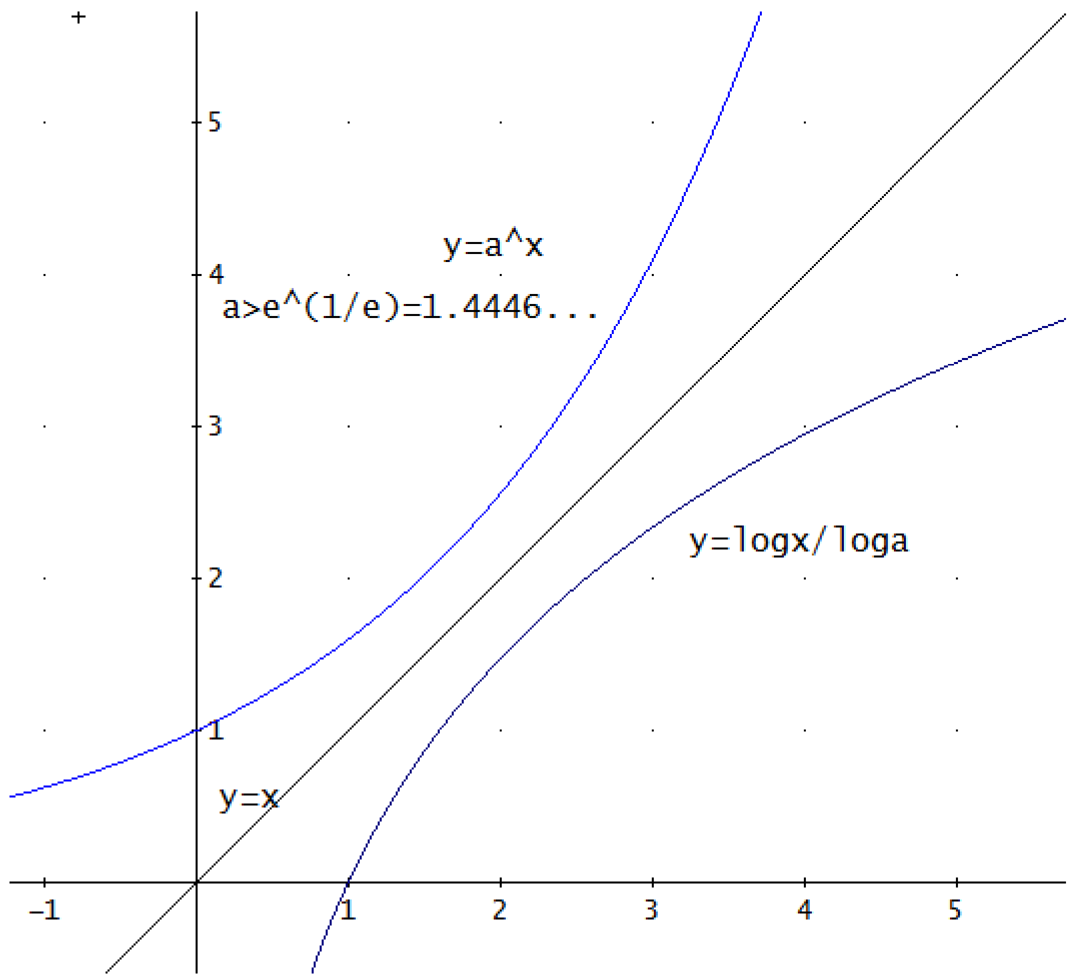


Case 3: For $a > e^{1/e}$ ($a = 1.6 > 1.4446\ldots$) there are no intersections

Figure 9: Graphs of $y = a^x$

$y = \log_a x$ and $y = x$

for $a = 1.6$



Calculus Classroom Capsules
Session S186

References

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