Learning Challenges and Teaching Strategies for Series in Calculus

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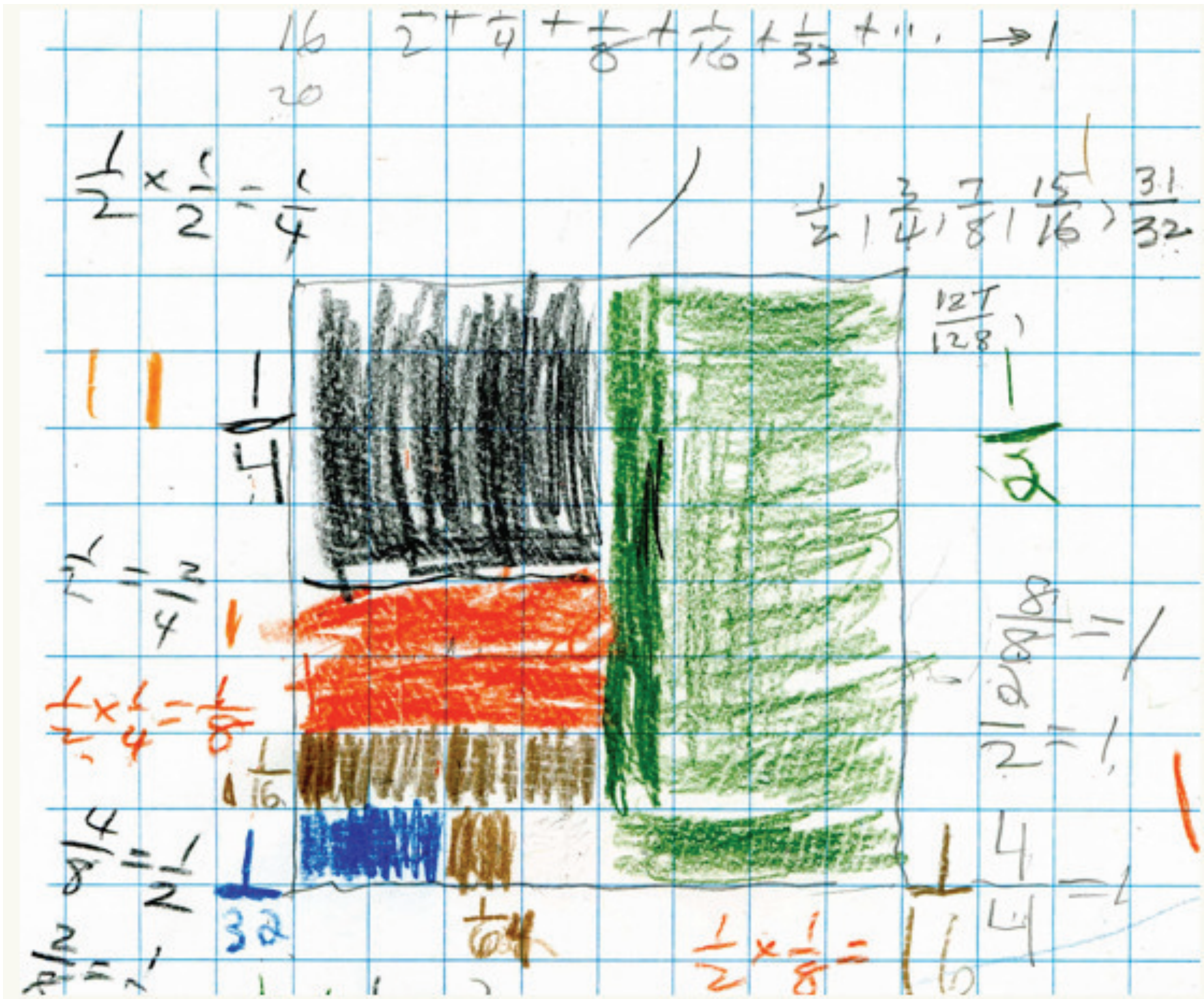
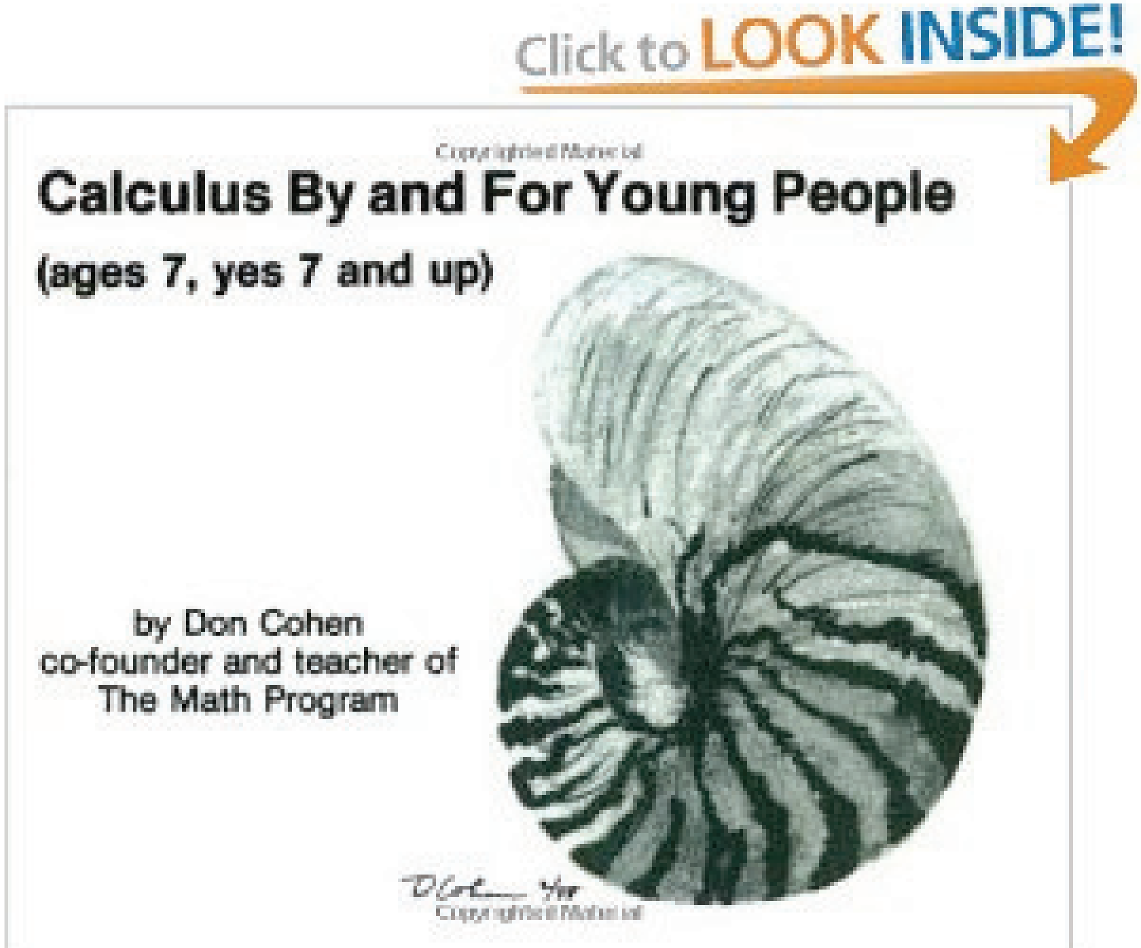


Ian, a second grader working with “Mathman” Don Cohen. www.mathman.biz



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Calculus By and For Young People
(ages 7, yes 7 and up)
by Don Cohen
co-founder and teacher of
The Math Program
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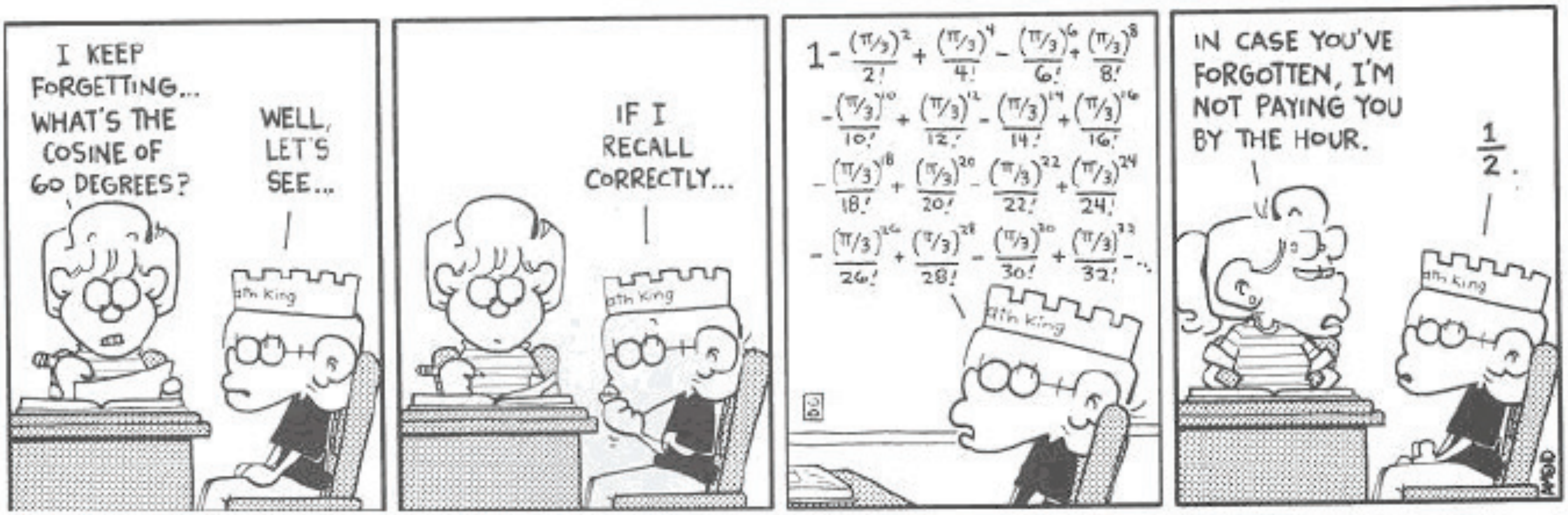

I KEEP FORGETTING... WHAT'S THE COSINE OF 60 DEGREES?
WELL, LET'S SEE...
IF I RECALL CORRECTLY...
$1-\frac{(\pi/3)^2}{2!}+\frac{(\pi/3)^4}{4!}-\frac{(\pi/3)^6}{6!}+\frac{(\pi/3)^8}{8!}-\frac{(\pi/3)^{10}}{10!}+\frac{(\pi/3)^{12}}{12!}-\frac{(\pi/3)^{14}}{14!}+\frac{(\pi/3)^{16}}{16!}-\frac{(\pi/3)^{18}}{18!}+\frac{(\pi/3)^{20}}{20!}-\frac{(\pi/3)^{22}}{22!}+\frac{(\pi/3)^{24}}{24!}-\frac{(\pi/3)^{26}}{26!}+\frac{(\pi/3)^{28}}{28!}-\frac{(\pi/3)^{30}}{30!}+\frac{(\pi/3)^{32}}{32!}-\ldots$
IN CASE YOU'VE FORGOTTEN, I'M NOT PAYING YOU BY THE HOUR.
$\frac{1}{2}$.

An infinite crowd of mathematicians enters a bar. The first one orders a pint, the second one a half pint, the third one a quarter pint...

"I understand", says the bartender - and pours two pints.

http://www.math.ualberta.ca/~runde/jokes.html

Jacob Bernoulli

Just as a finite sum confines an infinite series
And in what has no bounds there's still a bound,
So traces of divine immensity adhere to bodies
Of lowly kind, whose narrow bounds yet have no bound.
What a delight to spot the small in vast expanses,
To spot in smallness, what a joy, the immense God!

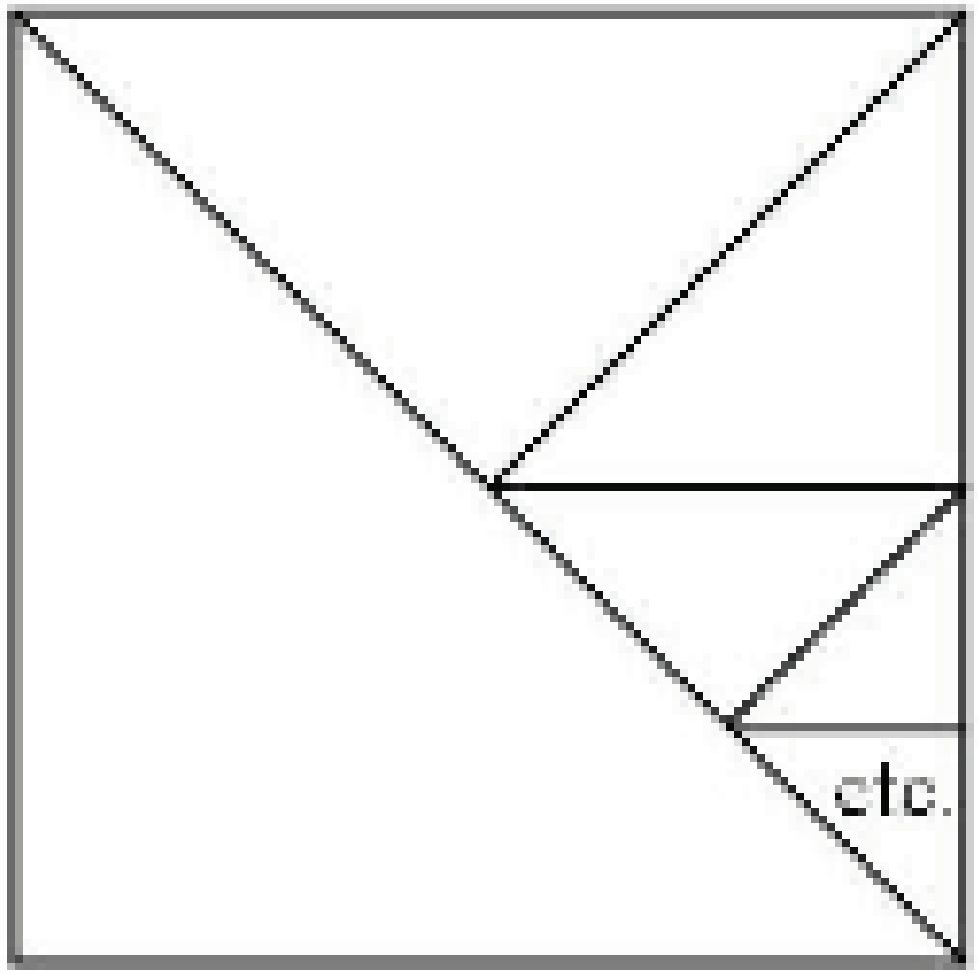
Written in Latin. Translated by Martin Mattmuller, 2009.

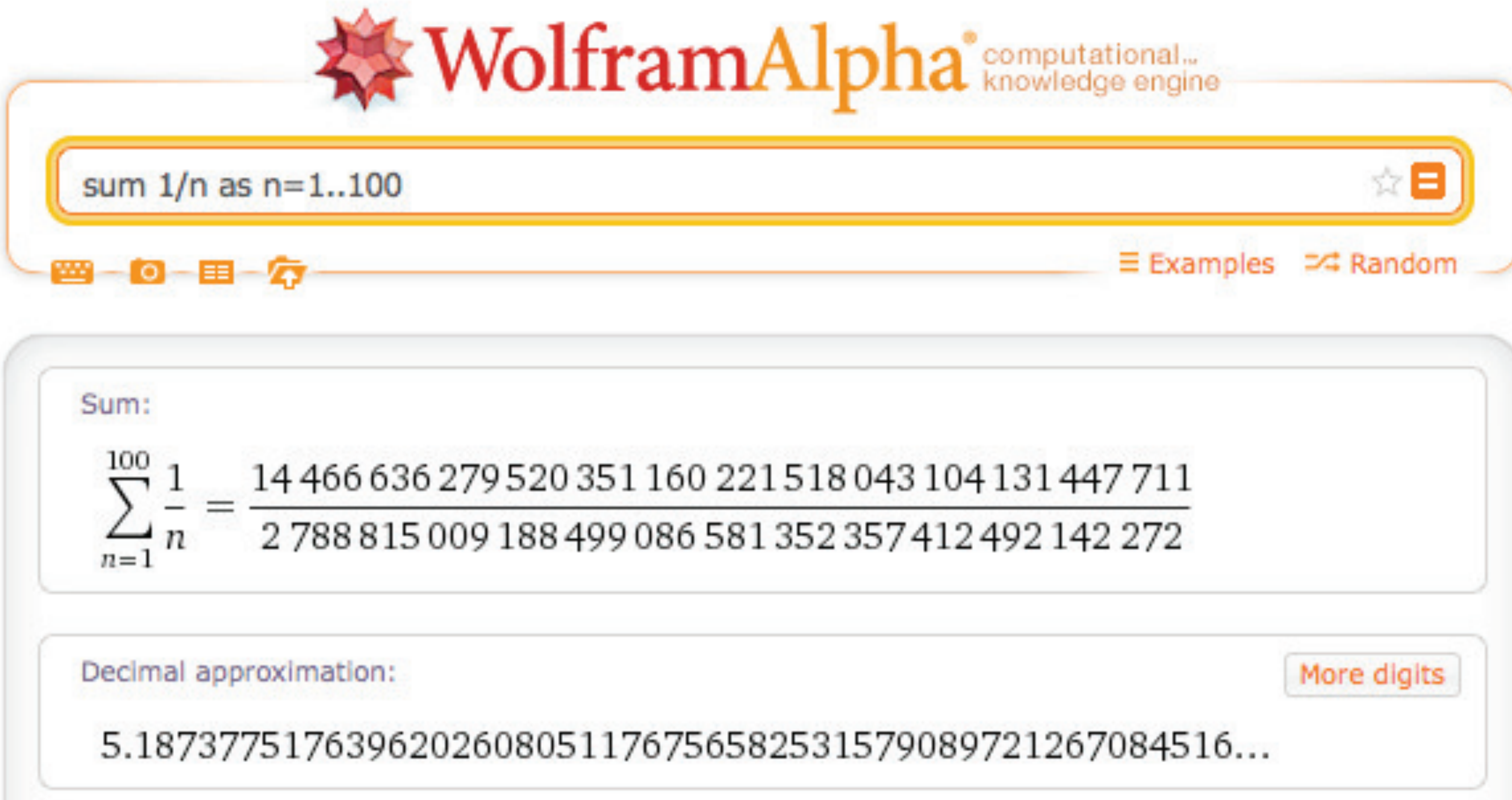
Zeno's Paradox

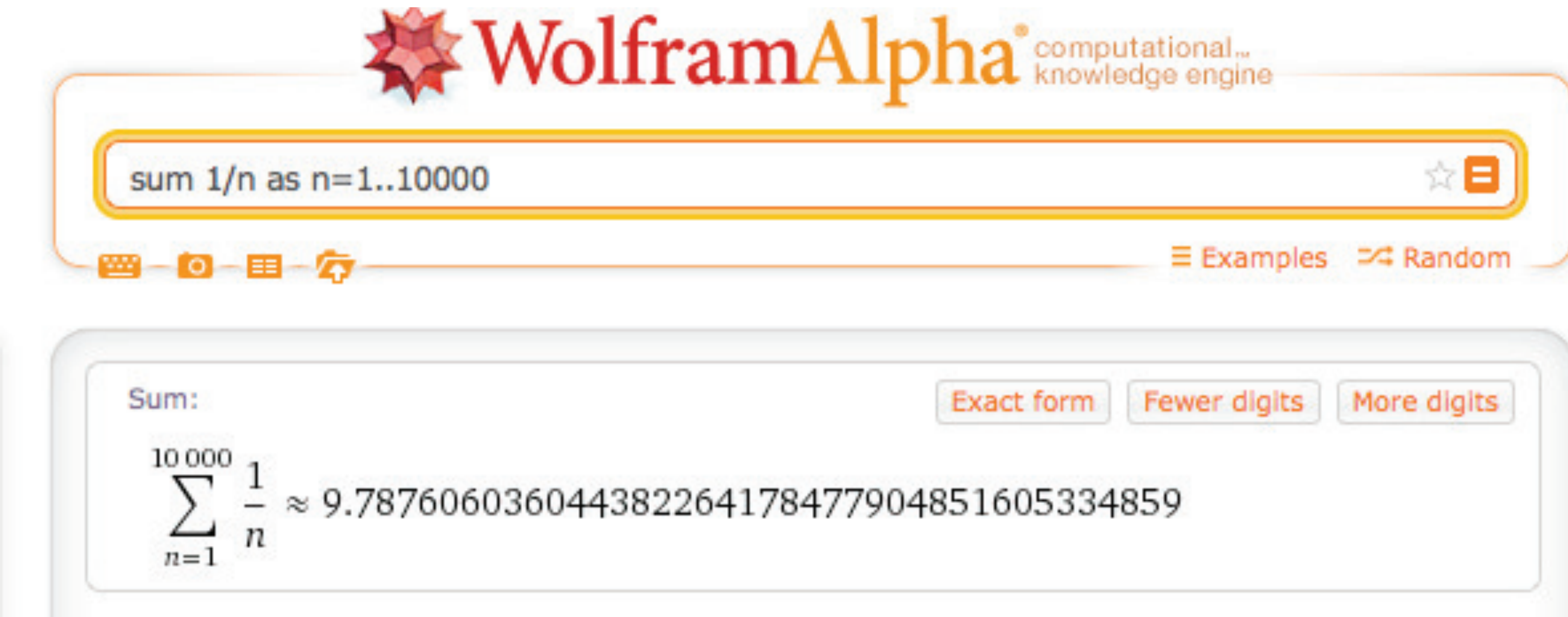
- That which is in locomotion must arrive at the half-way stage before it arrives at the goal.
  - Aristotle, Physics VI:9

Zeno's Infinite Pizza

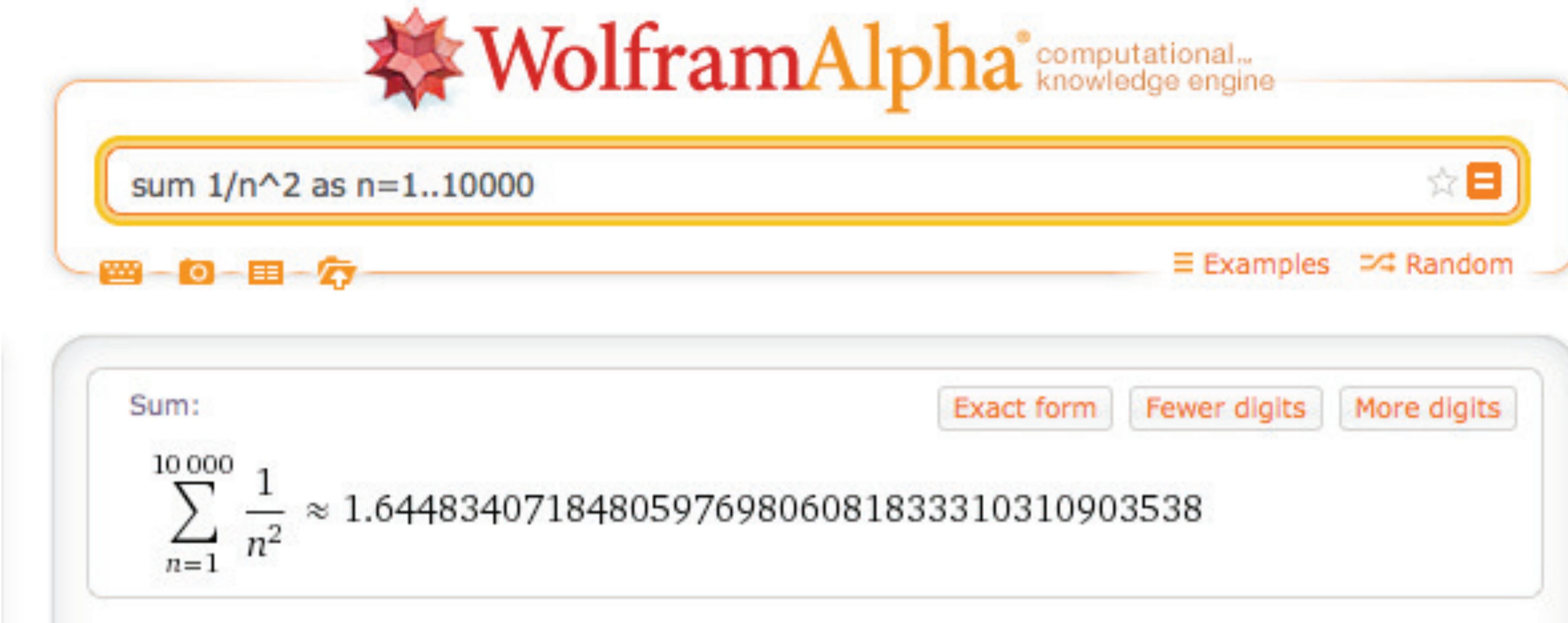
- Day 1: Zeno eats half the pizza so half remains.
- Day 2: Zeno eats half of what is left so 1/4 remains.
- Day 3: Zeno eats half of what is left so 1/8 remains.
- Day n: Zeno eats half of what is left so 1/(2^n) remains.


etc.


WolframAlpha computational knowledge engine
sum 1/n as n=1..100
Examples
Random
Sum:
$$\sum_{n=1}^{100} \frac{1}{n} = \frac{14\,466\,636\,279\,520\,351\,160\,221\,518\,043\,104\,131\,447\,711}{2\,788\,815\,009\,188\,499\,086\,581\,352\,357\,412\,492\,142\,272}$$
Decimal approximation:
More digits
5.1873775176396202608051176756582531579089721267084516…


WolframAlpha computational knowledge engine
sum 1/n as n=1..10000
Examples
Random
Sum:
Exact form
Fewer digits
More digits
$\sum_{n=1}^{10\,000} \frac{1}{n} \approx 9.787606036044382264178477904851605334859$


WolframAlpha computational knowledge engine
sum 1/n as n=1..10000000
Examples
Random
Approximated sum:
More digits
$\sum_{n=1}^{10\,000\,000} \frac{1}{n} \approx 16.6953$


WolframAlpha
computational...
knowledge engine
sum 1/n^2 as n=1..10000
Examples
Random
Sum:
Exact form
Fewer digits
More digits
$\sum_{n=1}^{10\,000} \frac{1}{n^2} \approx 1.6448340718480597698060818333103109035 38$


WolframAlpha computational knowledge engine
sum 1/n^2 as n=1..100000
Examples
Random
Approximated sum:
More digits
$$\sum_{n=1}^{100\,000} \frac{1}{n^2} \approx 1.64492$$


WolframAlpha computational knowledge engine
sum 1/n^2 as n=1..infinity
Examples
Random
Infinite sum:
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Question

Does $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ or is $\sum_{n=1}^{\infty} \frac{1}{2^n} < 1$?

Concepts needed to answer the question.

- Partial sum definition of convergence of a series.
- Epsilon-delta definition of convergence of a sequence.
- Students struggle with both of these concepts.

Equivalent Questions

- Does 0.333333… = 1/3?
- Does 0.999999… = 1?
- Do geometric series converge?

Geometric Series

$$\sum_{n=0}^{\infty} a \cdot r^n = a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

$$S = a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

$$rS = ar + ar^2 + ar^3 + ar^4 + \cdots$$

$$S - rS = a$$

$$S(1 - r) = a$$

BE CAREFUL!

$$S = \frac{a}{1 - r}$$

Counter Example

$$S = 1 + 2 + 4 + 8 + 16 + 32 + \cdots$$

$$2S = 2 + 4 + 8 + 16 + 32 + \cdots$$

$$S - 2S = 1$$

$$-S = 1$$

$$S = -1$$

$$1 + 2 + 4 + 8 + 16 + 32 + \cdots = -1$$

Partial Sums

$$S_1 = 1$$

$$S_2 = 3$$

$$S_3 = 7$$

$$S_4 = 15$$

$$S_n = 2^n - 1$$

$$\lim_{n \to \infty} S_n \text{ does not exist}$$

Telescoping Series

$$\sum_{n=0}^{\infty} \frac{4}{n^2+4n+3} = \sum_{n=0}^{\infty} \frac{4}{(n+1)(n+3)} = \sum_{n=0}^{\infty} \left[\frac{2}{(n+1)} - \frac{2}{(n+3)} \right]$$

$$\sum_{n=0}^{\infty}\frac{4}{n^2+4n+3}=\sum_{n=0}^{\infty}\frac{4}{(n+1)(n+3)}=\sum_{n=0}^{\infty}\left[\frac{2}{(n+1)}-\frac{2}{(n+3)}\right]$$

$$S=\left[\frac{2}{1}-\frac{2}{3}\right]+\left[\frac{2}{2}-\frac{2}{4}\right]+\left[\frac{2}{3}-\frac{2}{5}\right]+\left[\frac{2}{4}-\frac{2}{6}\right]+\left[\frac{2}{5}-\frac{2}{7}\right]+\cdots$$

$$S=\left[\frac{2}{1}-\frac{\mathbf{2}}{\mathbf{3}}\right]+\left[\frac{2}{2}-\frac{\mathbf{2}}{\mathbf{4}}\right]+\left[\frac{\mathbf{2}}{\mathbf{3}}-\frac{\mathbf{2}}{\mathbf{5}}\right]+\left[\frac{\mathbf{2}}{\mathbf{4}}-\frac{2}{6}\right]+\left[\frac{\mathbf{2}}{\mathbf{5}}-\frac{2}{7}\right]+\cdots$$

$$S=2+1=3$$

Partial Sums

$$S_2 = 3 - \frac{2}{3} - \frac{2}{4}$$

$$S_3 = 3 - \frac{2}{4} - \frac{2}{5}$$

$$S_4 = 3 - \frac{2}{5} - \frac{2}{6}$$

$$S_n = 3 - \frac{2}{n+1} - \frac{2}{n+2} = 3 - \frac{4n+6}{n^2+3n+2}$$

$$\lim_{n \to \infty} S_n = 3 - 0 = 3$$

Why do We Study Infinite Series?

$$5^{2/3} = \sqrt[3]{5^2}$$

$$5^{\pi} = ?$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$5^\pi = e^{\pi \ln 5} = 1 + (\pi \ln 5) + \frac{(\pi \ln 5)^2}{2!} + \frac{(\pi \ln 5)^3}{3!} + \frac{(\pi \ln 5)^4}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{(\pi \ln 5)^n}{n!}$$

Euler's Rule

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Why?

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

$$e^{i\theta} = 1 + (i\theta) - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \frac{(\theta)^4}{4!} + \frac{i(\theta)^5}{5!} - \frac{(\theta)^6}{6!} + \cdots$$

Series are the “DNA” of Transcendental Functions

- They are used by calculators to determine values of transcendental functions.
- They are used to solve difficult limits and integrals.
- They are used to find approximate solutions.

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$$

k	
10	3.23232
100	3.15149
1000	3.142592
10000	3.141692

$$\int_0^{1/2} \frac{1}{1+x^3}\, dx$$

$$\frac{1}{1+x^3} = \frac{a_0}{1-r}$$

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + x^{12} - x^{15} + \cdots$$

$$\int_0^{1/2} \frac{1}{1+x^3}\, dx = \int_0^{1/2} \left[1 - x^3 + x^6 - x^9 + x^{12} - x^{15} + \cdots\right] dx$$

$$= \left[x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \frac{x^{13}}{13} - \frac{x^{16}}{16} + \cdots\right]_0^{1/2} \quad = 0.48540185$$

Oresme ca. 1350

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\cdots$$

$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+$$

$$\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)+\cdots$$

$$\geq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

Historical Perspectives

- Aristotle - Physics, III, IV, 206b, 1-33.
- Archimedes - Quadratura parabolae
- Andreas Tacquet (1612-1660)
- Guido Grandi (1671-1742): 1-1+1-1+… = 1/2
- Leibniz (1646-1716) argued using probability to support Grandi's claim.
- Riccati (1676-1754) argued against Grandi
- Gauss (1777-1855) defined convergence "correctly."

G. Bagni: University of Udine

http://www.math.wpi.edu/IQP/BVCalcHist/calc3.html

Basel Problem

Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$

Euler solved it in 1735. Jakob Bernoulli, Leibniz and Mengoli tried to solve the problem but they were unsuccessful.

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Open Problem

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

http://plus.maths.org/content/infinite-series-surprises

Literature

- Students believe that 0.9999999…<1, yet they correctly evaluate $\sum_{n=1}^{\infty} \frac{9}{10^n}$

Artigue et al. (2007)

Roh (2005)
Common misconceptions

- a series continues endlessly so it has no limit
- a limit can be found by plugging infinity in for n and evaluating algebraically
- the series gets close to the number but never actually gets there

Roh

- the series needs to get close to a number or arrive at a number, thus resulting in two limits for a series
- a sequence has a limit if differences between consecutive terms get smaller

Alcock

- Students with an internal sense of authority tend to link their visual and symbolic representations of series. Those with an external sense of authority believe that written work is a collection of conventions and procedures , They see no reason to try to integrate visual and symbolic representations.

Hardy on Limits

- What students learn in a calculus class differs from what the instructor intended to teach. Rather than using concepts from calculus to solve limit problems, students tend to focus on practiced algebraic procedures used to solve typical exam questions.

Sequence and Series Concepts Inventory

- Define sequence as it is used in calculus.
  - 40% correct
- Construct an example of a sequence
  - 34% correct
- What does it mean for a sequence to converge?
  - 57% correct
- Define series as it is used in calculus.
  - 46% correct

- Construct an example of a series.
  – 51% correct
- What does it mean for a series to converge?
  – 40% correct
- How does a series differ from a sequence?
  – 46% correct
- Construct an example of a series that converges to 4.
  – 23% correct

- Construct an example of a sequence that diverges.
  - 29% correct
- Construct an example of a series that converges to 5.
  - 14% correct
- Construct an example of a series that diverges.
  - 66% correct
- Describe the conditions and conclusions of the nth term test.
  - 29% correct

Types of Errors

- Algebraic Errors
- Calculus Conceptual Errors
- Incorrect Conditions or Conclusions
- Choosing the Incorrect Strategy
- Stating Facts without Proof
- Misunderstanding the Question or the Notation

Algebraic Errors

- Incorrect properties of exponents
- Incorrect properties of radicals
- Incorrect reduction of fractions
- Incorrect inequalities

Calculus Conceptual Errors

- Incorrect limit evaluation
- Incorrect use of L’Hopital’s rule
- Incorrect evaluation of an integral in the integral test.
- Incorrect properties of derivatives or integrals

Incorrect Conditions or Conclusions

- Using the integral test for a series that is not decreasing
- Using the LCT when the limit of an/bn is 0 or infinity
- Incorrect conclusion for ratio/root test
- Nth-term convergence test

Choosing the Incorrect Strategy

- Using DCT instead of LCT when inequalities fail
- Failing to recognize telescoping series
- Using p-series test when a series is geometric

Stating Facts without Proof

$$\sum_{n=1}^{\infty} \frac{5^n}{9^n - 4^n}$$

$$9^n - 4^n > 6^n$$

$$\frac{1}{9^n - 4^n} < \frac{1}{6^n}$$

$$\frac{5^n}{9^n - 4^n} < \frac{5^n}{6^n}$$

Converges by DCT

Misunderstanding Question or Notation

- Proving that a series converges when the question asks to find the sum
- Difficulty identifying the common ratio in a geometric series
- Ignoring index of a series

What is to be done?

- Ask non-traditional questions.
  - Construct three sequences that behave differently.
  - How does a series relate to a sequence?
  - If the sequence converges to 0, does the series converge?
  - What does it mean for a series to converge?

What else?

- Use technology to compute partial sums.
- Use graphical representations of sequences and series.
- Ask questions that challenge faulty concept images.
- Have students generate formal personal definitions of convergence.
- Use improper integrals to reinforce the notions of infinite series. (Fay, 1985)

Thank You!

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- Slides available
  sites.google.com/site/RWCAMATYC/files