Mathematical Modeling: The Right Courses for the Right Students for the Right Reasons

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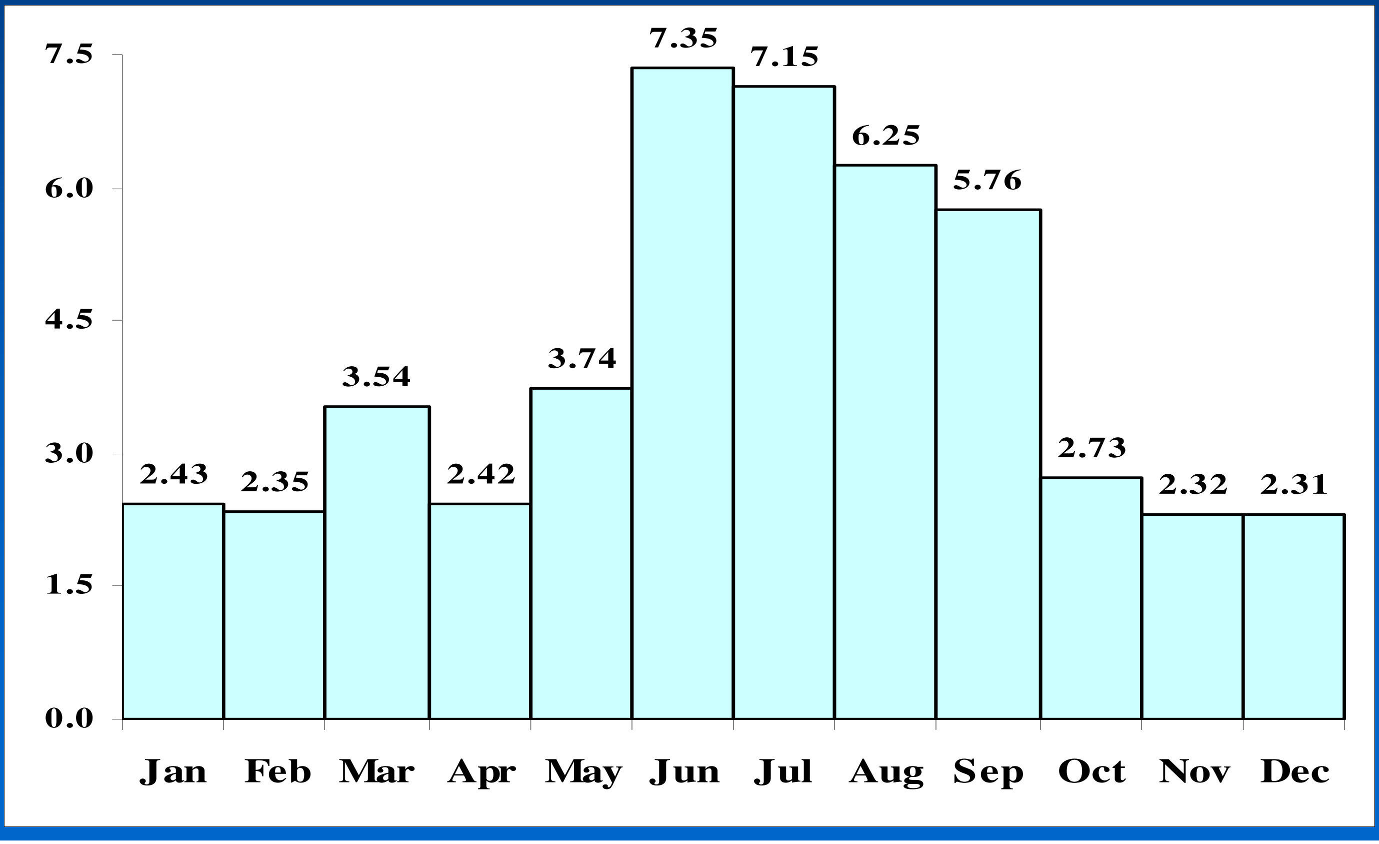
Accessing the Presentation

This PowerPoint presentation and the DIGMath Excel files that will be used can all be downloaded from:

farmingdale.edu/faculty/sheldon-gordon/

Monthly Rainfall in Orlando

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2.43	2.35	3.54	2.42	3.74	7.35	7.15	6.25	5.76	2.73	2.32	2.31



Average Worldwide Temps, °C

Year	1880	1900	1920	1940	1960	1980	1990	2000
Temp	13.80	13.95	13.90	14.15	14.00	14.20	14.40	14.50

(a) Decide which is the independent variable and which is the dependent variable.
(b) Decide on appropriate scales for the two variables for a scatterplot.
(c) State precisely which letters you will use for the two variables and state what each variable you use stands for.
(d) Draw the associated scatterplot.
(e) Raise some predictive questions in this context that could be answered when we have a formula relating the two variables.

A Modeling-Based Course: Topic 2

Behavior of functions as data and as graphs, including increasing/decreasing, turning points, concave up/down, inflection points (including the logistic function and the normal distribution function patterns).

Behavior of Functions

The graph shows the amount of profit from an investment over the course of 40 days after it was purchased.

a. On which days is the investment a gain?

b. On which days is the investment a loss?

c. On which intervals is the profit increasing?

d. On which intervals is the profit decreasing?

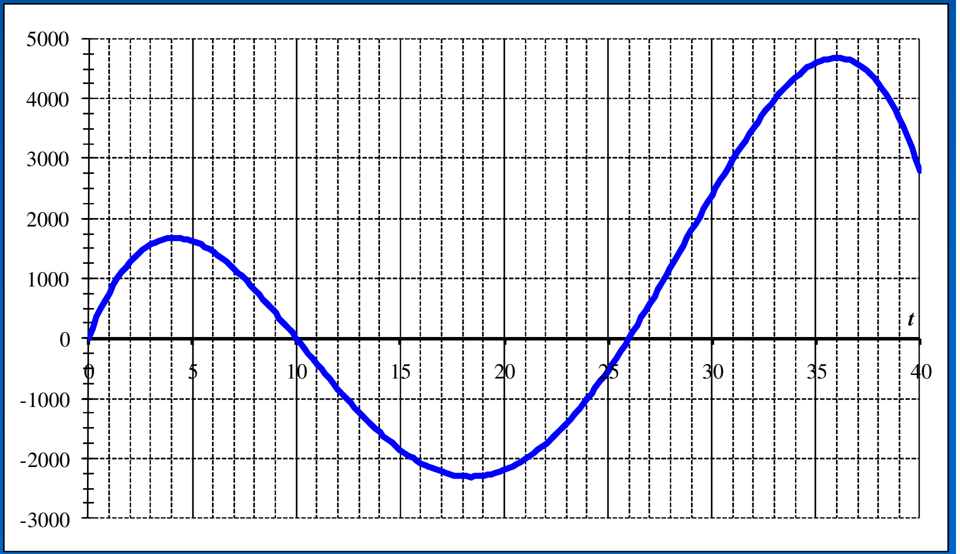
e. On which intervals is the profit function concave up?

f. On which intervals is the profit function concave down?

g. Estimate the coordinates of all turning points.

h. Estimate the coordinates of all inflection points.

i. If the investment was sold on the 40th day, estimate how much of a profit or loss resulted.



A Modeling-Based Course: Topic 3

Linear functions, with emphasis on the meaning of the parameters and fitting linear functions to data, including the linear correlation coefficient to measure how well the regression line fits the data.

Linear Functions

The snowy tree cricket, which lives in the Colorado Rockies, has a chirp rate, *R*, in chirps per minute, that is related to the temperature, *T*, in Fahrenheit.

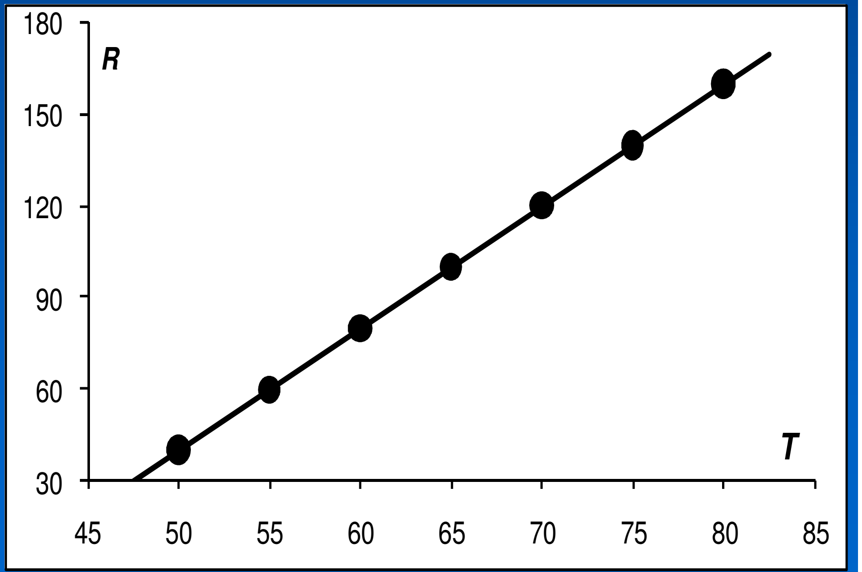
Temp	50	55	60	65	70	75	80
Chirp Rate	40	60	80	100	120	140	160

a. Find a linear function that models the chirp-rate as a function of temperature.

b. Discuss the practical meaning of the slope and the vertical intercept and give reasonable values for the domain and range.

c. Predict the chirp rate at a temperature of 84°.

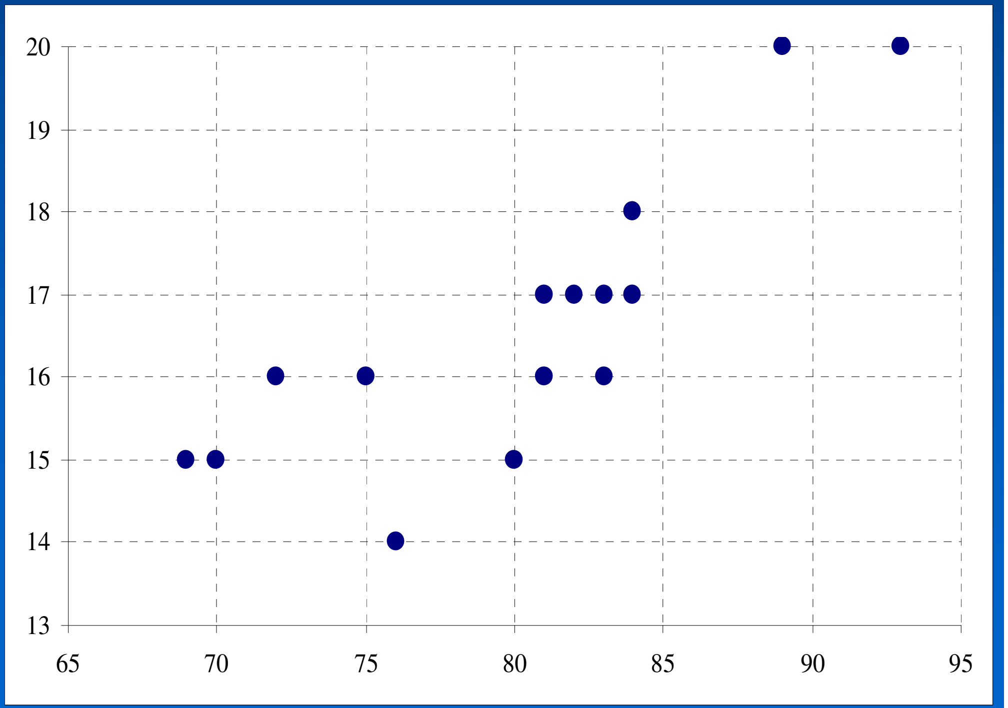
d. Predict the temperature if a cricket is chirping 138 times per minute.



Regression by hand

The following table gives measurements for the chirp rate (chirps per minute) of the striped ground cricket as a function of the temperature.

T	89	72	93	84	81	75	70	82
R	20	16	20	18	17	16	15	17
T	69	83	80	83	81	84	76	
R	15	16	15	17	16	17	14	

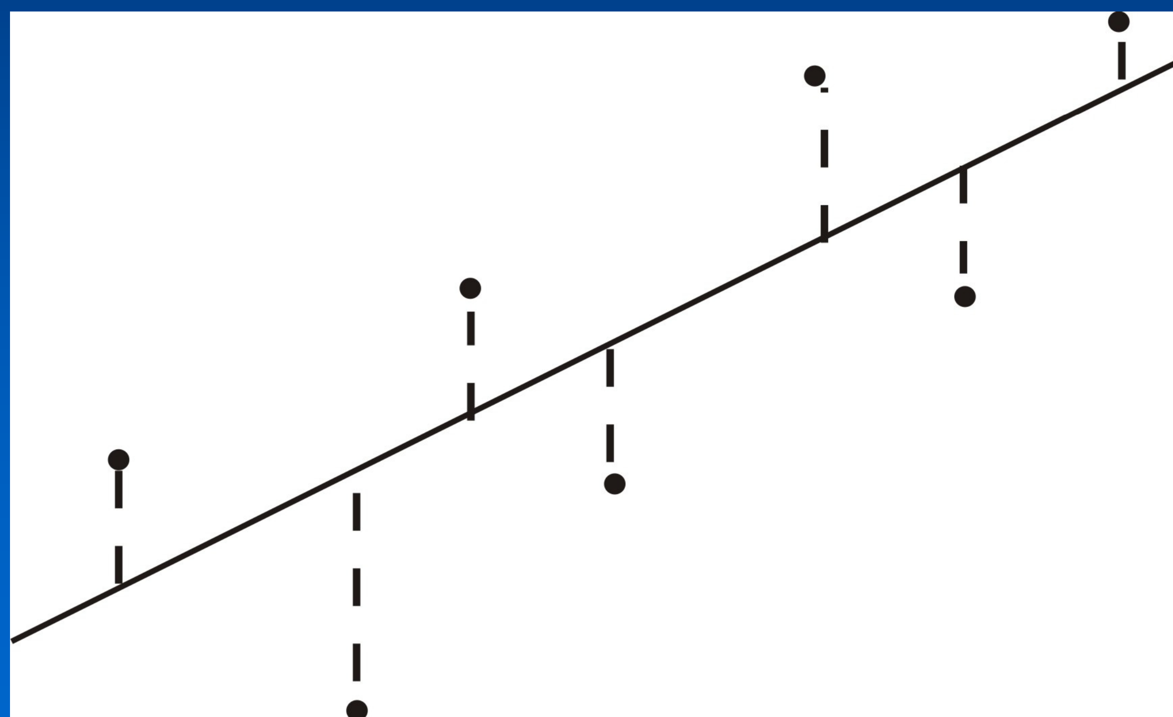


Regression Simulation

Regression using Calculators and/or Excel

The Sum of the Squares

The *regression line* is the line that comes *closest* to all of the data points in the *least-squares sense,* meaning that the sum of the squares of the vertical deviations is a minimum.



A Modeling-Based Course: Topic 4

Nonlinear families of functions:

- exponential growth and decay, applications such as population growth and decay of a drug in the body; doubling time and half-life;
- power functions;
- logarithmic functions;
- Fitting each family of functions to data based on the behavioral characteristics of the functions and deciding on how good the fit is.

Families of Functions

After linear functions, the most common and useful families of functions in terms of realistic applications and models are, in order:

1. exponential functions
2. power functions
3. logarithmic functions
4. sinusoidal functions
5. polynomial functions

In contrast, in traditional mathematics, the major emphasis is on polynomials because they provide so many opportunities to practice algebraic skills.

Exponential Functions

For linear functions, the successive differences are constant. That constant difference is related to the slope of the line.

For exponential functions $y = A\,b^x$, the successive ratios are constant.

When the ratio of successive values b is greater than 1, it is an exponential growth function and b is the *growth factor*.

When the ratio of successive values b is less than 1, it is an exponential decay function and b is the *decay factor*.

Modeling Drug Levels in the Blood

Every drug is washed out of the bloodstream, usually by the kidneys, though at a different rate that is characteristic of the particular drug.

For example, in any 24-hour period, about 25% of any Prozac in the blood is washed out, leaving 75% of the amount.

Modeling Drug Levels in the Blood

$D_0 = 80$ mg.

After 24 hours (1 day), 25% is removed, leaving

$D_1 = 0.75D_0 = 60$ mg.

After another day, 25% of D_1 is removed, leaving

$D_2 = 0.75D_1 = 0.75(0.75D_0) = (0.75)^2 D_0 = 45$ mg.

After another day, 25% of D_2 is removed, leaving

$D_3 = 0.75D_2 = 0.75(0.75)^2 D_0 = (0.75)^3 D_0 = 33.75$ mg.

In general, after n days, the level of Prozac is:

$$D_n = (0.75)^n D_0 = 80(0.75)^n$$

Predictive Questions

- **What will the level of the drug be after a given number of time periods?**

- **How long will it take until the drug level is down to a given level?**

How long until Prozac level is 10 mg.?

Solve: $80(0.75)^n = 10$

Need to use logs to solve algebraically.

An Exponential Decay Model

When a person smokes a cigarette, about 0.4 mg of nicotine is absorbed into the blood. Each hour, about 35% of any nicotine present is washed out of the blood.

a. Write the equation of a function that models the level of nicotine in the blood after a single cigarette.

b. Use your model to estimate how long it takes for the amount of nicotine to drop to 0.005 mg.

Doubling Time & Half-Life

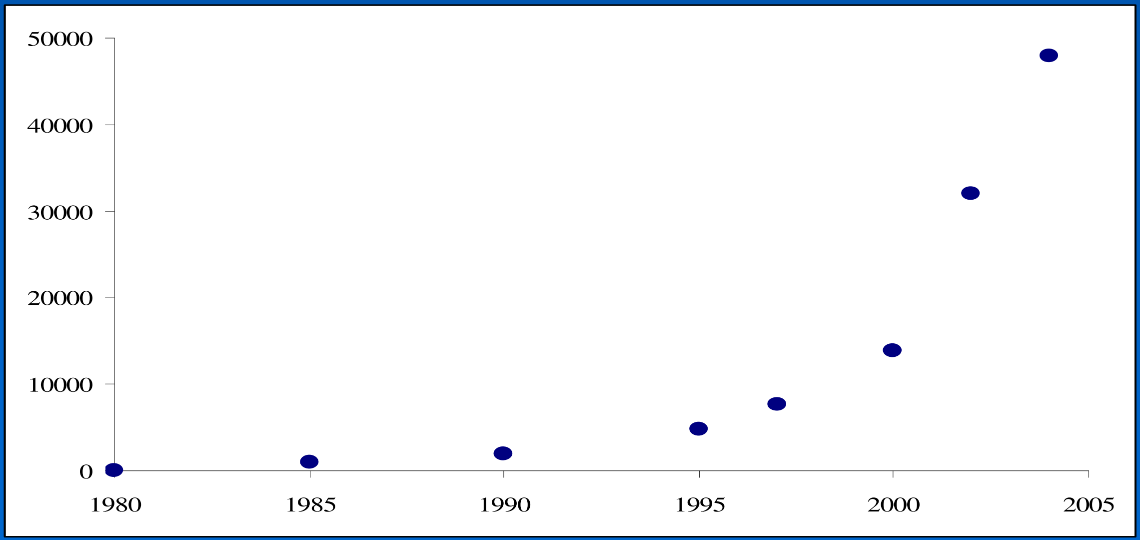
Every exponential growth function $y = A\,b^t$, $b > 1$, has a characteristic *doubling time* – the length of time needed for it to double in size. The doubling time depends *only* on the growth factor b.

Every exponential decay function $y = A\,b^t$, $0 < b < 1$, has a characteristic *half-life* – the length of time needed for it to decrease by half. The half-life depends *only* on the decay factor b.

Fitting Exponential Functions

The following table shows world-wide wind power generating capacity, in megawatts, in various years.

Year	1980	1985	1990	1995	1997	2000	2002	2004
Wind power	10	1020	1930	4820	7640	13840	32040	47910



(a) Which variable is the independent variable and which is the dependent variable?
(b) Explain why an exponential function is the best model to use for this data.
(c) Find an exponential function that models the relationship between the amount of power *P* generated by wind and the year *t*.
(d) What are some reasonable values that you can use for the domain and range of this function?
(e) What is the practical significance of the base (1.1373) in the exponential function you created in part (c)?
(f) What is the doubling time for this function? Explain what it means. Solve: $52.497(1.1373)^t = 2\times 52.497$.
(g) According to your model, what do you predict for the total wind power generating capacity in 2012?

How Exponential Regression Works

If a set of (x, y) data follows an exponential pattern, then the transformed $(x, \log y)$ data follow a linear pattern.

Method Used:

Transform the data to $\log y$ vs. x

Fit a linear function to the transformed data.

Undo the transformation to create the exponential function.

Undoing the Semi-log Transformation

The linear fit to the (log *W*, *t*) data for wind power is:

$$y = 0.1298x + 1.6897 \quad \text{or} \quad \log W = 0.1298t + 1.6897$$

To undo the logs, we take powers of 10:

$$\begin{aligned} 10^{\log W} = W &= 10^{0.1298t + 1.6897} \\ &= 10^{0.1298t} \cdot 10^{1.6897} \\ &= 48.944 \cdot (10^{0.1298})^t \\ &= 48.944 \cdot (1.3483)^t, \end{aligned}$$

Wind power has been growing at a rate of 34.83% a year.

How Non-Linear Fits Are Done

If a set of (x, y) data follows an exponential pattern, then the transformed (x, log y) data follow a linear pattern.

If a set of (x, y) data follows a power function pattern, then the (log x, log y) data follow a linear pattern.

If a set of (x, y) data follows a logarithmic pattern, then the (log x, y) data follow a linear pattern.

Correlation with Non-Linear Fits

In the process, a value for the linear correlation coefficient *r* is calculated. It only measures how well the linear function fits the transformed data, not how well the nonlinear function fits the original data. It is this value that is reported by Excel and by graphing calculators. As such, it must be interpreted with care – students need to realize that it does not measure how well the nonlinear function fits the data.

How Good is the Fit?

How does one measure how well a particular function actually fits a set of data?

How does one decide on which of several possible candidates is the “best” fit to a set of data?

1. You could, but probably shouldn’t, use only the correlation coefficient.
2. You could use the sum of the squares to measure and compare different fits.
3. You can “eyeball” how well the function appears to fit the data.

Where Do You Find Data?

Statistical Abstract of the U.S. (on-line)

www.census.gov/statab/www/

***Vital signs: The Trends that are Affecting Our Lives*. (annual book published by the WorldWatch Foundation)**

***On Size and Life*, Scientific American Lib.**

***Functions, Data, and Models: An Applied Approach to College Algebra*, Gordon & Gordon, MAA Textbooks.**

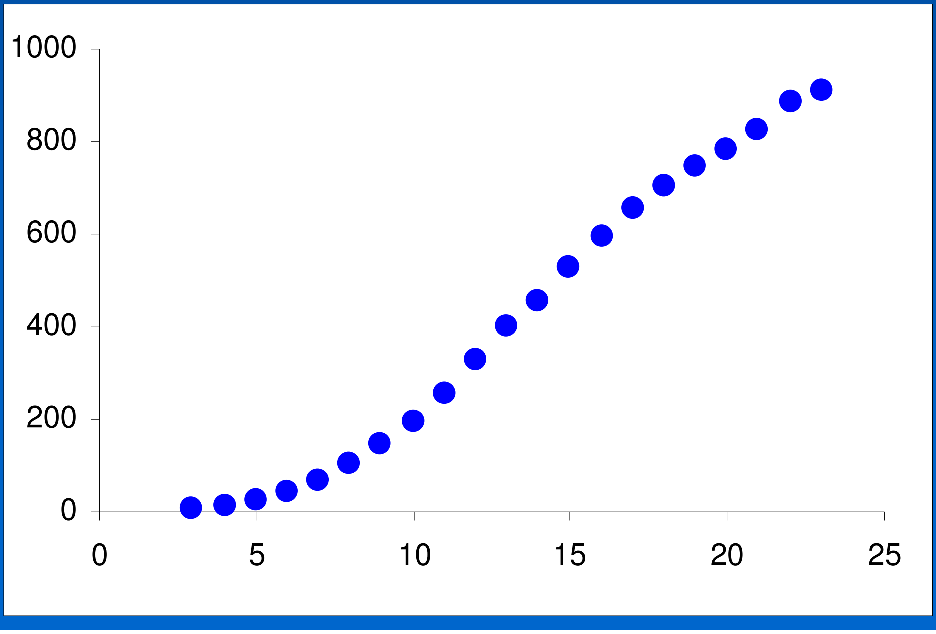
A Modeling-Based Course: Topic 5

Modeling with Polynomial Functions: Emphasis on the behavior of polynomials and modeling, primarily by fitting polynomials to data

Modeling the Spread of AIDS

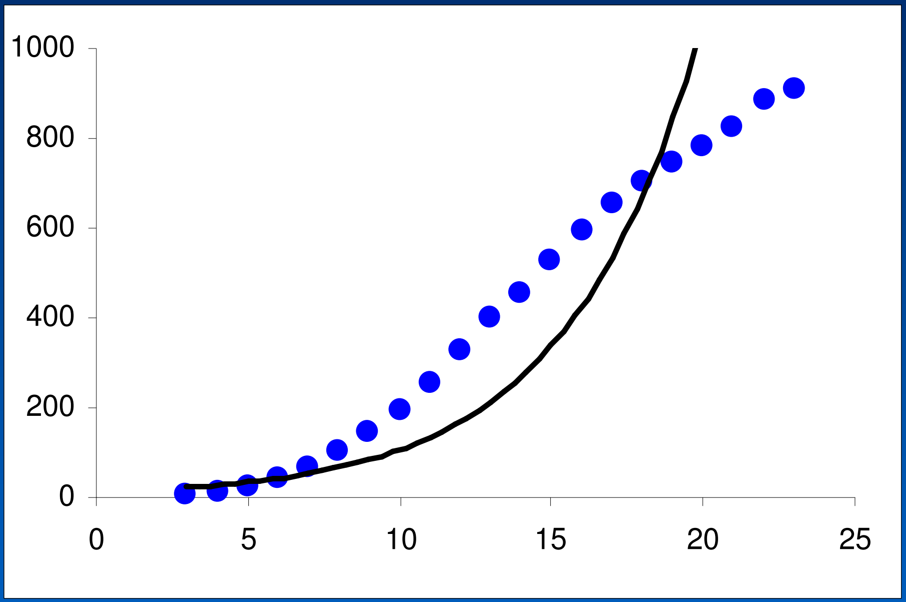
The total number of reported cases of AIDS in the United States since it was first diagnosed in 1983:

Year	1983	1984	1985	1986	1987	1988	1989	1990
AIDS Cases	4,589	10,750	22,399	41,256	69,592	104,644	146,574	193,878
Year	1991	1992	1993	1994	1995	1996	1997	1998
AIDS Cases	251,638	326,648	399,613	457,280	528,144	594,641	653,084	701,353
Year	1999	2000	2001	2002	2003			
AIDS Cases	742,709	783,976	824,809	886,098	909,269			



Modeling the Spread of AIDS

An Exponential Function:



$$A = 10.742(1.2583)^t,$$

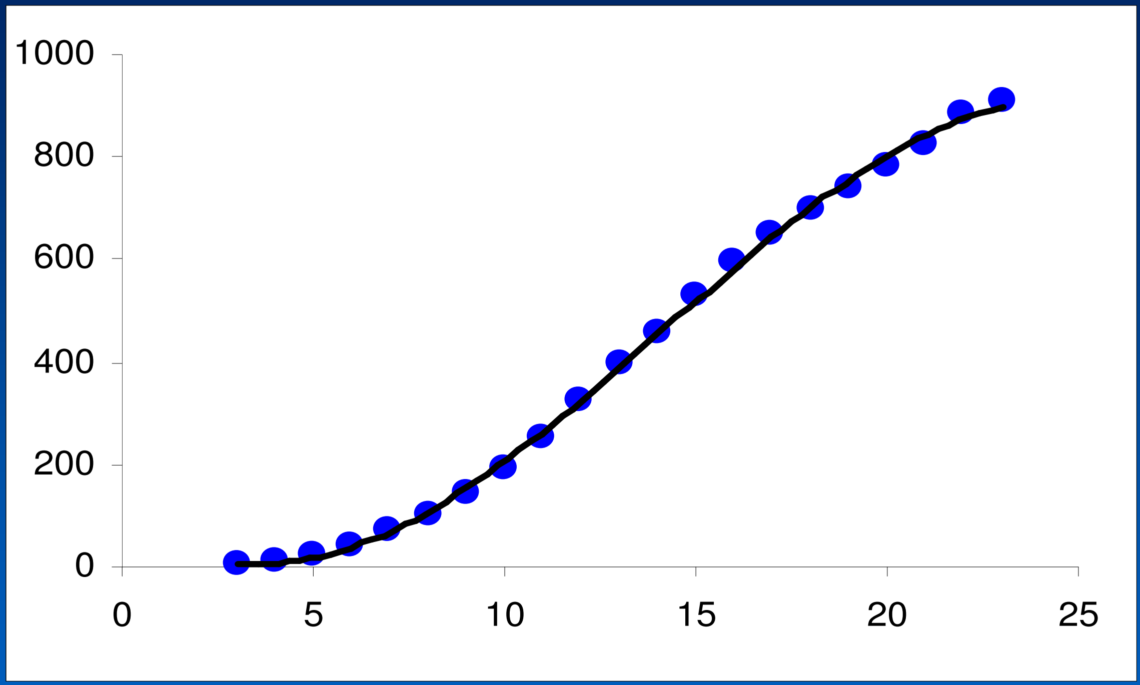
where t is measured in years since 1980.

The correlation coefficient $r = 0.9176$.

The Sum of the Squares = 2,655,781.

Modeling the Spread of AIDS

A cubic polynomial:



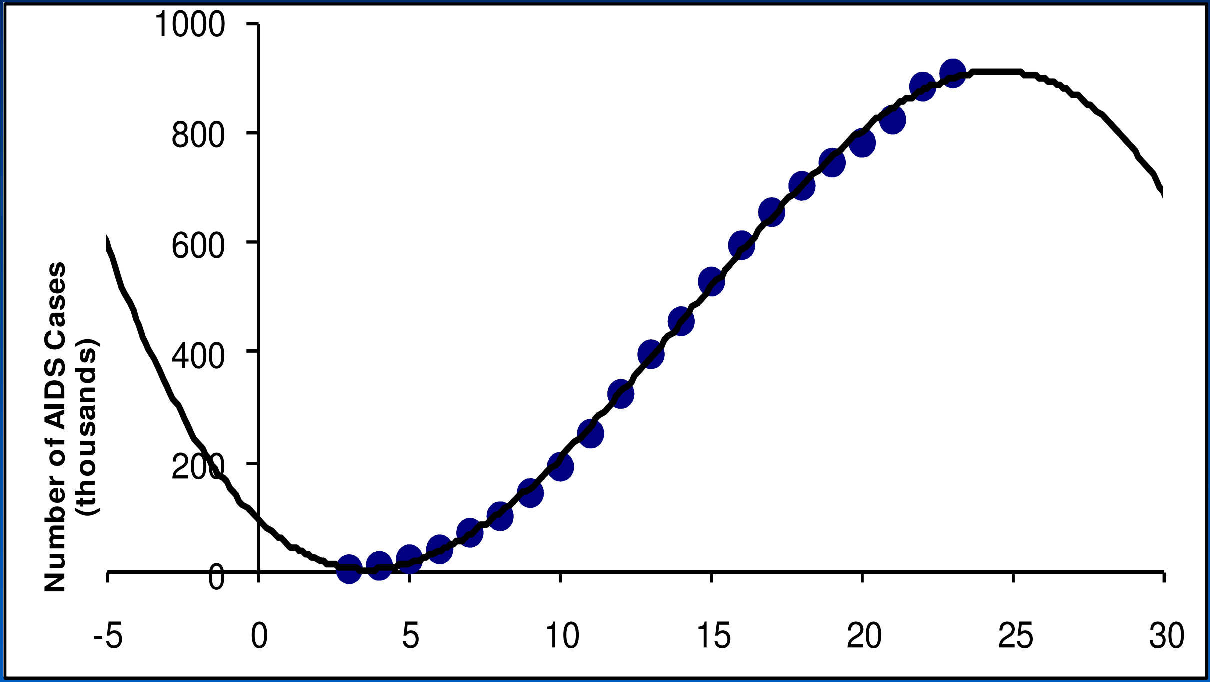
$$A = -0.20t^3 + 8.39t^2 - 53.23t + 97.97,$$

where t is the number of years since 1980.

The Sum of the Squares = 3,489.

Modeling the Spread of AIDS

A larger window:



$$A = -0.20t^3 + 8.39t^2 - 53.23t + 97.97,$$

where t is the number of years since 1980.

Notice that he leading coefficient is negative.

A Modeling-Based Course: Topic 8

Sinusoidal Functions and Periodic Behavior: Using the sine and cosine as models for periodic phenomena such as the number of hours of daylight, heights of tides, average temperatures over the year, etc.

Jacksonville Daylight Function

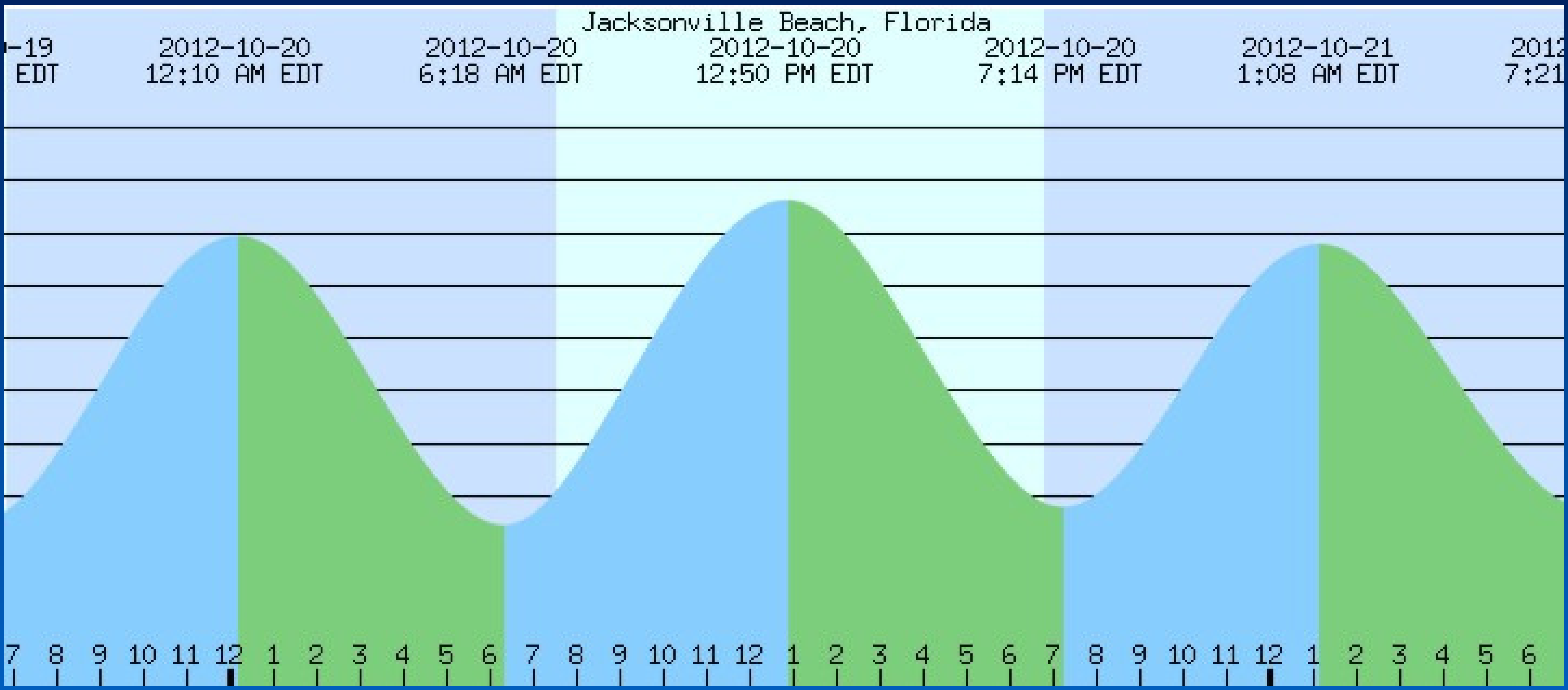
$$H = 12 + 1.48\sin\left(\frac{2\pi}{365}(t-80)\right)$$

What do the four numbers represent?

- **What does the 365 represent in practice?**
- **What does the 12 represent in practice?**
- **What does the 1.48 represent in practice?**
- **What does the 80 represent in practice?**

Sinusoidal Functions

Sinusoidal Functions



Sinusoidal Functions

Using such displays, one can ask:

1. Estimate the period.
2. Estimate the maximum and minimum; estimate the midline.
3. Estimate the amplitude.
4. What is the frequency?
5. Estimate the phase shift for a cosine or sine function.
6. Write a formula to model the phenomenon.

Dallas Temperatures

The table gives the average daytime high temperature in Dallas roughly every 2 weeks.

Day	1	15	32	46	60	74	91	105	121	135	152	196	213
Temp	55	53	56	59	63	67	72	77	81	84	89	98	99

Day	227	244	258	274	288	305	319	335	349
Temp	98	94	90	85	80	72	66	61	58

Create a sinusoidal function to model the average daytime high temperature in Dallas.

The values in the table are temperatures roughly every 2 weeks, but two entries are missing. Predict the average daytime high temperature in Dallas on those dates.

The Sum of the Squares

Can you get a better fit based on the sum of the squares?

This is a lovely classroom investigation in which students suggest minor changes to the values of the parameters and immediately see whether they can improve on the fit.

In the process, they get a much deeper understanding of the meaning of each of the parameters.

The average daytime high temperature in New York as a function of the day of the year varies between 32°F and 94°F. Assume the coldest day occurs on the 30^{th} day and the hottest day on the 214^{th}.

a. Sketch the graph of the temperature as a function of time over a three year time span.

b. Write a formula for a sinusoidal function that models the temperature over the course of a year.

c. What are the domain and range for this function?

d. What are the amplitude, vertical shift, period, frequency, and phase shift of this function?

e. Predict the most likely high temperature on March 15.

f. What are all the dates on which the high temperature is most likely 80°?

The air conditioner in a home is set to come on when the temperature reaches 76 ° and to turn off when the temperature drops to 72 °. This cycle repeats every 20 minutes, starting at noon.

a. Sketch the graph of the temperature as a function of time over a one hour time span. (Assume the temperature is 76 ° at the start, which is at noon.)

b. Write a formula for a sinusoidal function that models the temperature over the course of time.

c. What are the domain and range for this function?

d. What are the amplitude, vertical shift, period, frequency, and phase shift of this function?

e. Predict the temperature at 12:08; at 12:26.

f. What are all the times between noon and 1 pm when temperature is 73°?

For More Information

This PowerPoint presentation and the DIGMath Excel files demonstrated can all be downloaded:

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