What's an Inflection Point?

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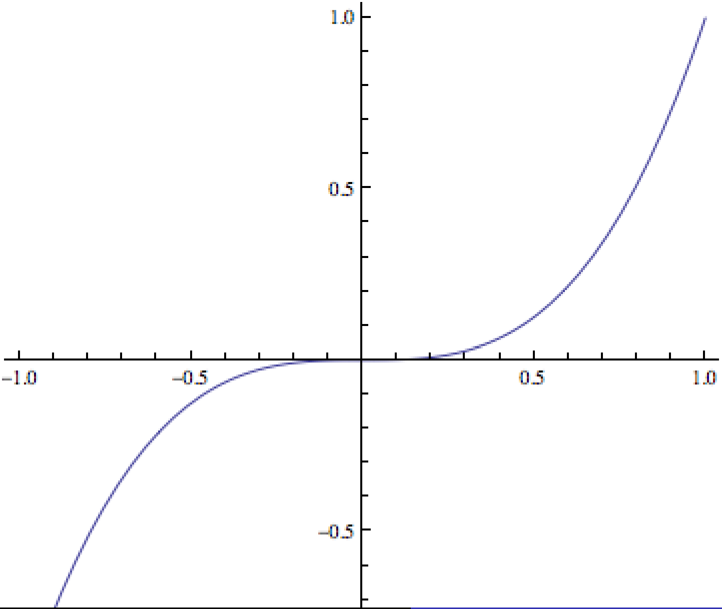
November 10, 2012

Is $(0, 0)$ an inflection point?

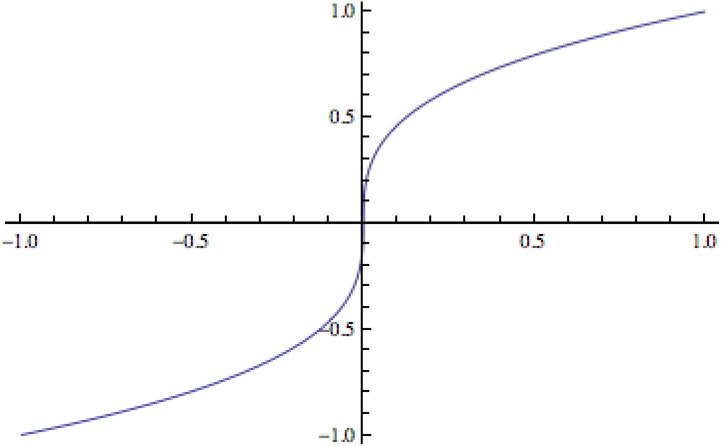
Curve 1

$y = x^3$



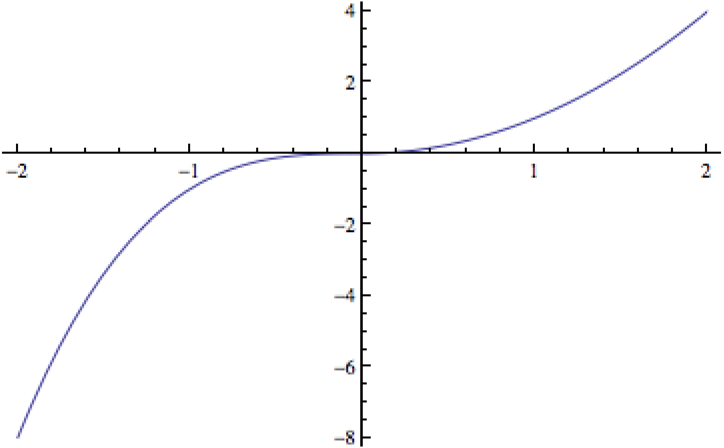
Curve 2

$y = \sqrt[3]{x}$



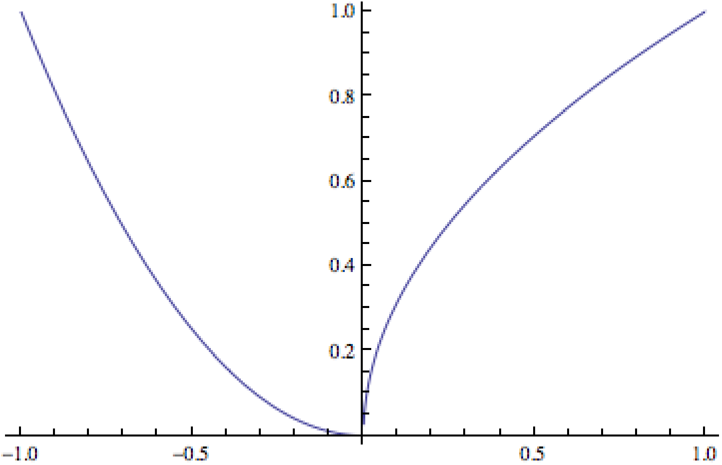
Curve 3

$$y = \begin{cases} x^3, & \text{if } x < 0 \\ x^2, & \text{if } x \geq 0 \end{cases}$$



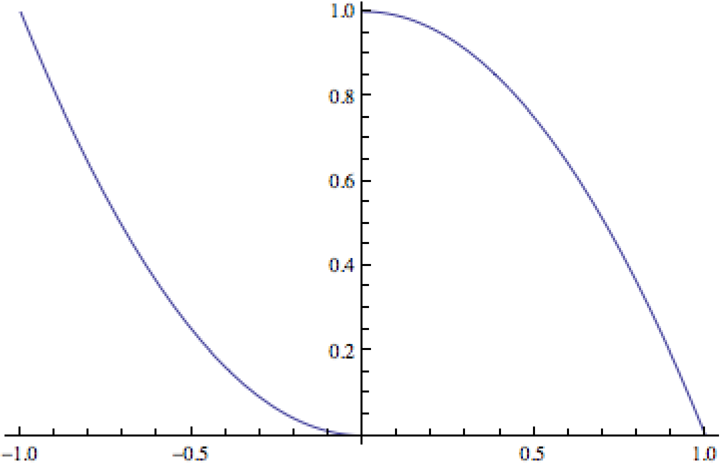
Curve 4

$$y = \begin{cases} x^2, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } x \geq 0 \end{cases}$$



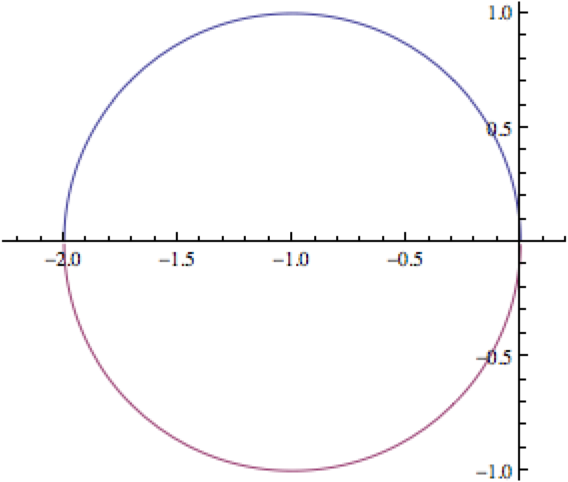
Curve 5

$$y = \begin{cases} x^2, & \text{if } x \le 0 \\ -x^2 + 1, & \text{if } x > 0 \end{cases}$$



Curve 6

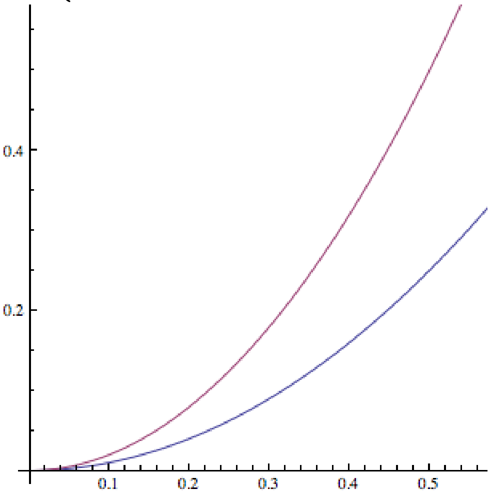
$(x+1)^2 + y^2 = 1$



Curve 7

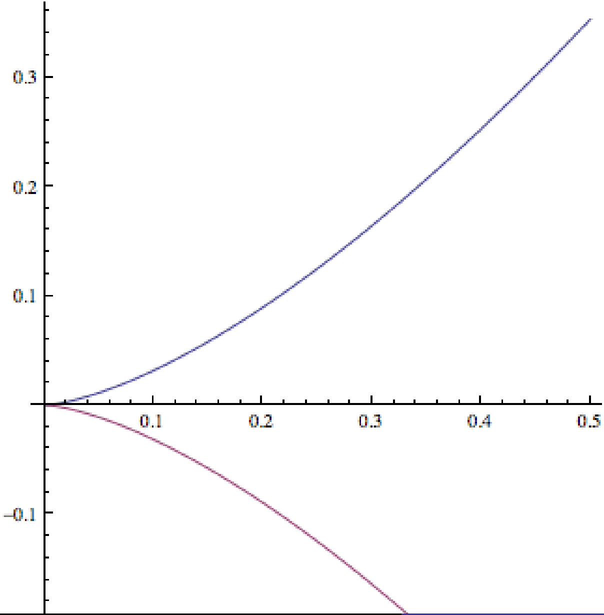
$x = t^2$

$$y = \begin{cases} t^4, & \text{if } t < 0 \\ 2t^4, & \text{if } t \geq 0 \end{cases}$$



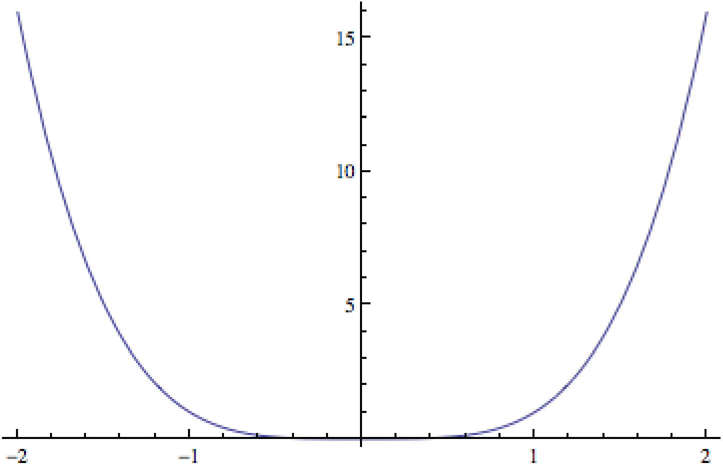
Curve 8

$x = y^{2/3}$



Curve 9

$y = x^4$



Common Definitions

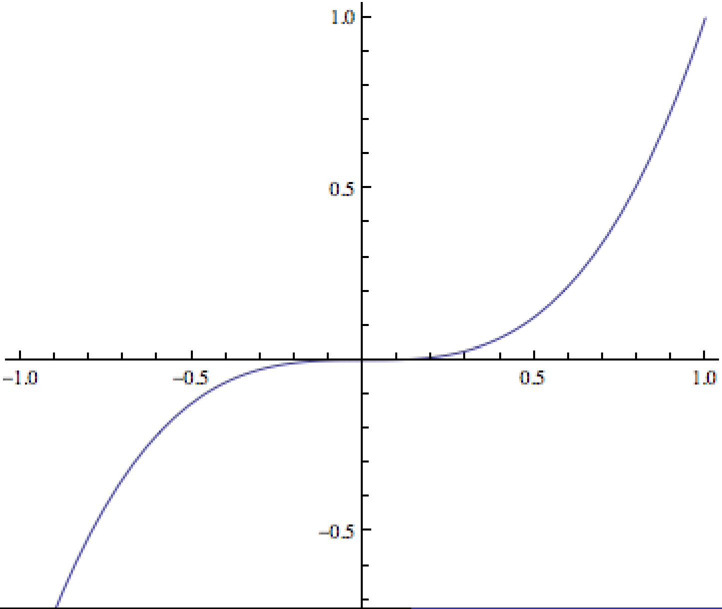
Definition A: concavity changes

Definition B: f' attains a maximum or minimum

Definition C: tangent line crosses the curve

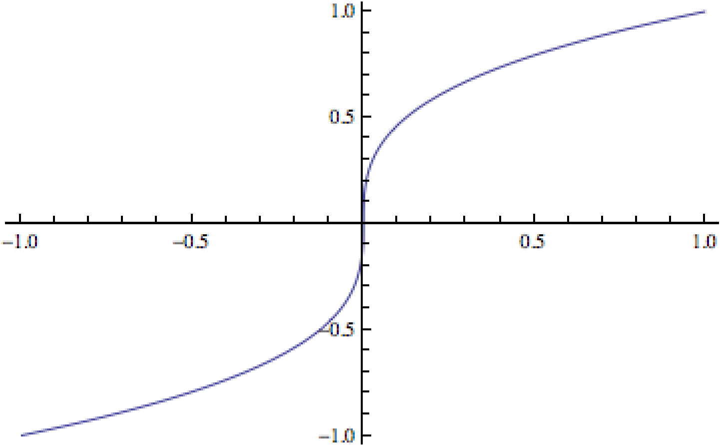
Curve 1 satisfies definitions A, B, C

$y = x^3$



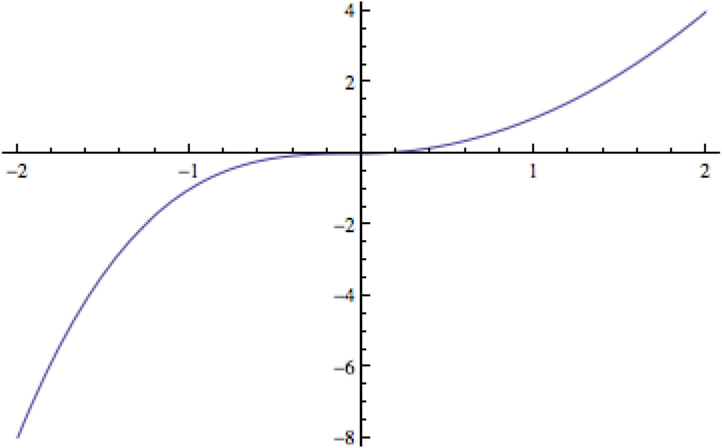
Curve 2 satisfies definitions A, C

$y = \sqrt[3]{x}$



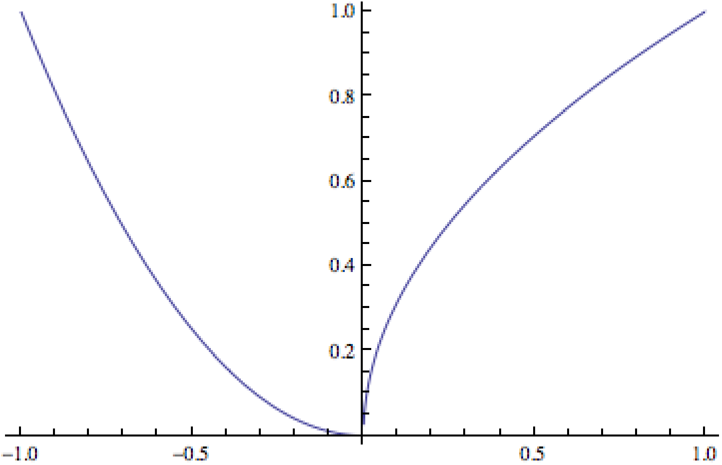
Curve 3 satisfies definitions A, B, C

$$y = \begin{cases} x^3, & \text{if } x < 0 \\ x^2, & \text{if } x \geq 0 \end{cases}$$



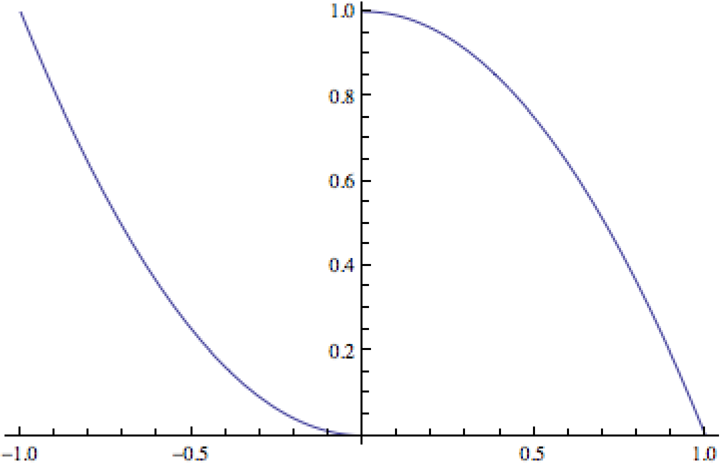
Curve 4 satisfies definition A

$$y = \begin{cases} x^2, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } x \geq 0 \end{cases}$$



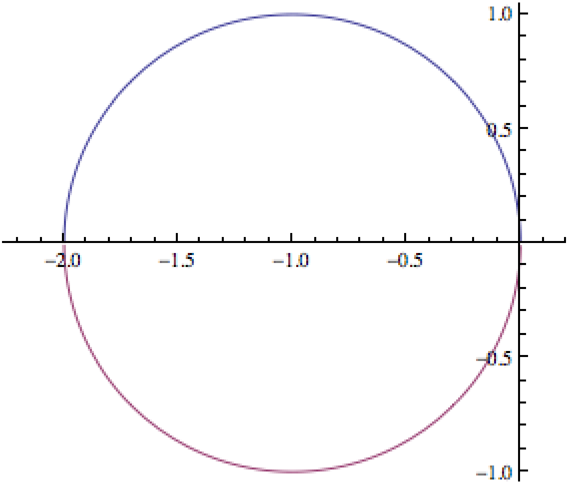
Curve 5 satisfies definition A

$$y = \begin{cases} x^2, & \text{if } x \leq 0 \\ -x^2 + 1, & \text{if } x > 0 \end{cases}$$



Curve 6 satisfies definition A

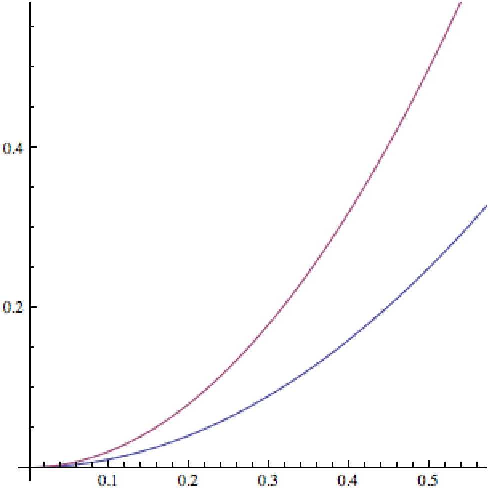
$(x+1)^2 + y^2 = 1$



Curve 7 satisfies definition B

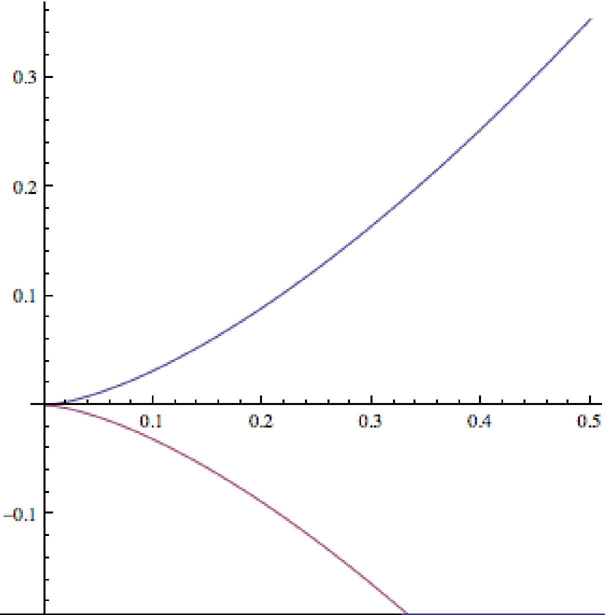
$x = t^2$

$$y = \begin{cases} t^4, & \text{if } t < 0 \\ 2t^4, & \text{if } t \geq 0 \end{cases}$$



Curve 8 satisfies definitions A, C

$x = y^{2/3}$



Ewing's definition (1938):
An inflection point is a point $(x(t), y(t))$ of a curve $C : x = x(\overline{t}); y = y(\overline{t})$ if and only if the following conditions are satisfied:

(1) $x(t)$ and $y(t)$ are continuous for $t = \overline{t}$

(2) there exist numbers $t_1 < \overline{t}$ and $t_2 > \overline{t}$ such that $D(\tau_1, \tau_2, \tau_3) > 0$ (or < 0) for $t_1 \leq \tau_1 < \tau_2 < \tau_3 \leq \overline{t}$, and $D(\tau_1, \tau_2, \tau_3) < 0$ (or > 0 respectively) for $\overline{t} \leq \tau_1 < \tau_2 < \tau_3 \leq t_2$

(3) for every linear form $ax + by + c$ which vanishes for $x = x(\overline{t}), y = y(\overline{t})$ there exist numbers $t_3 < \overline{t}$ and $t_4 > \overline{t}$ such that $ax(t) + by(t) + c > 0$ (or < 0 respectively) for $t_3 \leq t < \overline{t}$ and $ax(t) + by(t) + c < 0$ (or > 0 respectively) for $\overline{t} < t \leq t_4$.

In other words, an inflection point is a point of a curve such that

(1') the curve is continuous at the point

(2') the direction of the concavity changes

(3') every straight line through the point has an arc of the curve on each side of it.

A.M. Bruckner (1962)

If f is differentiable on (a,b), then:

i.) if concavity of f changes at x_0 (def. A), then f' attains a max or min at x_0 (def. B).

ii.) if f' attains a max or min at x_0 (def. B), then the tangent line to f at x_0 crosses the graph (def. C).

A.M. Bruckner (1962)

i.) There exist functions with infinitely many derivatives which satisfy Definition C (f crosses tangent line) at x_0 but not Definition B (f' max or min) at x_0.

ii.) There exist functions with infinitely many derivatives which satisfy Definition B (f' max or min) at x_0 but not Definition A (concavity changes) at x_0.

A.M. Bruckner (1962)

i.)

$$f(x) = \begin{cases} (e^{-x^{-2}} \sin 1/x)^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -(e^{-x^{-2}} \sin 1/x)^2 & \text{if } x < 0 \end{cases}$$

f crosses its tangent line at $(0, 0)$ but f' does not attain a max or min at $(0, 0)$

A.M. Bruckner (1962)

ii.)

$$g(x) = \int_0^x h(t)dt$$

where $h(0) = 0$ and

$$h(t) = (e^{-x^{-2}} \sin 1/x)^2$$

$g' \geq 0$ and $g'(0) = 0$ but concavity doesn't change at $(0, 0)$ since g'' is both positive and negative in every interval $(a, 0)$ and $(0, b)$

A.M. Bruckner (1962)

f is analytic implies these three definitions are equivalent.

A real function is said to be analytic if it possesses derivatives of all orders *and* agrees with its Taylor series in a neighborhood of every point.

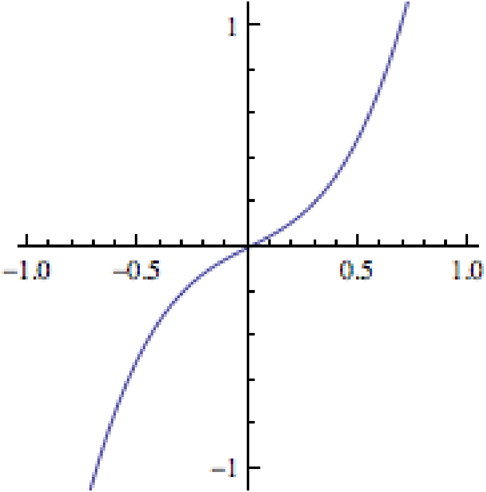
A.M. Bruckner (1962)

"Standard test for inflection points"

The first $n-1$ derivatives are 0 at the point, but the n^{th} derivative is non-zero and n is odd, $n \geq 3$.

G. H. Hardy's definition

$y = 2x^3 + .5x$



A.W. Walker (1956)

"The generally accepted view is surely ... that a point of inflection is any point where a plane curve crosses both its tangent and its normal."

A.W. Walker (1956)

The first $n-1$ derivatives are 0 at the point, but the n^{th} derivative is non-zero.

$n=3$: "ordinary" inflection point

$n>3$: "higher order" inflection point

$n=4$: undulation point

The history of the term "inflection point"

Fermat (1630's) "puncta inflexionum"

Huygens (1650's) "punctum flectionis contrarii"

Newton (1671) "punctum flexus contrarii"

Leibniz (1684) "punctum flexus contrarii"

Thomas Mulcrone (1968) "flexpoint"

Mulcrone (1968)

Authors using "flexpoint" (or some variation)

Davies and Peck (1869)
Winger (1923)
Cohen (1925)
Slobin and Solt (1935)
Murnaghan (1947)
James and James (1949)
R.J. Walker (1950)
Morril (1956)
Agnew (1962)
Bell, et al (1966)

Hergert Numbers

Where $f''(x) = 0$ or $f''(x)$ is undefined

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