For the Love of Linear Programming

A presentation in the field of mathematics known as Linear Programming

Presented by

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Linear Programming

A Little Bit of History

Definition

- Linear programming is a case of mathematical programming
- A technique for optimizing a linear function, subject to linear equality and linear inequality constraints
- The expression to be maximized or minimized is called the objective function
- The inequalities are the constraints which specify a feasible region over which the objective function is to be optimized
- Non-negative inequalities ensure that variables are positive

Founders

- The problem of solving a system of linear inequalities dates back at least as far as Fourier
- Leonid Kantorovich developed the earliest linear programming problems in 1939 for use during World War II to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy.
- The method was kept secret until 1947 when George B. Dantzig published the simplex method
- John von Neumann established the theory of duality that same year

Founders

- The Nobel prize in economics was awarded in 1975 to the mathematician Leonid Kantorovich (USSR) and the economist Tjalling Koopmans (USA) for their contributions to the theory of optimal allocation of resources
- The problems were first shown to be solvable in polynomial time by Leonid Khachiyan in 1979
- A larger theoretical and practical breakthrough in the field came in 1984 when Narendra Karmarkar introduced a new interior-point method

Linear Programming

What is it used for?

Uses

- Many practical problems in operations research can be modeled as linear programming problems
- Linear programming inspired many of the central concepts of optimization theory
- It is used in microeconomics and company management, such as planning, production, transportation, technology and other issues
- Modern management issues are ever-changing, but most companies want to maximize profits or minimize costs with limited resources

Linear Programming

Maximization of Profit
An Example

Example

Cell Phone Cover Manufacturing – The Percival Phone Protector Company makes two types of cell phone covers. It produces an essential cover (EC) that sells for a $25 profit and an essential deluxe cover (EDC) that sells for a $40 profit. On the manufacturing line the EC requires 2 hours and the EDC takes 3 hours. The finishing area spends 1 hour on the EC and 2 hours on the EDC. Both covers require 1 hour for testing and packing. On a particular production run, the Percival Company has at most 900 work hours on the manufacturing line, 550 work hours in the finishing area, and 400 work hours in the testing and packing area. How many phone covers of each type should be produced to maximize profit? What is the maximum profit?

Example

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Organize in a Table

Cover Type	Manufacturing	Finishing	Testing and Packing	Profit
Essential cover (EC)→ x_1	2	1	1	$25
Essential deluxe cover (EDC)→ x_2	3	2	1	$40
Total	≤ 900	≤ 550	≤ 400	

Mathematical Model

Maximize $P = 25x_1 + 40x_2$

subject to:

$$2x_1 + 3x_2 \le 900$$
$$x_1 + 2x_2 \le 550$$
$$x_1 + x_2 \le 400$$
$$x_1, x_2 \ge 0$$

This is termed

A Standard Maximization Problem in Standard Form

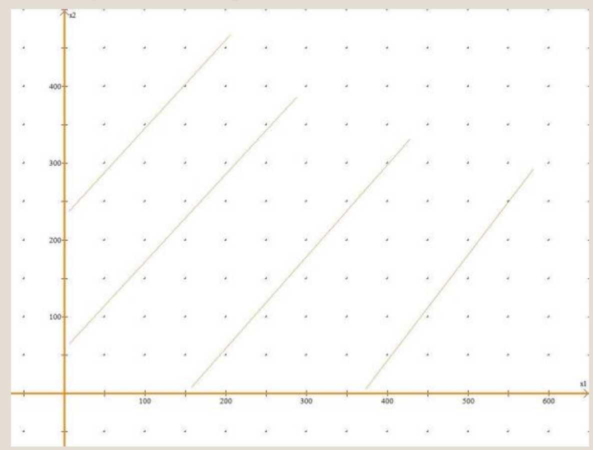
Linear Programming

Solving the Problem
Method 1 – Graphically

Graphically Solve

- The linear inequalities and non-negative constraints are graphed on the x-y axis; graph the line and then shade the region that gives a true statement
- The intersection of the shaded regions define the feasible region
- This feasible region can be termed as bounded (the region can be enclosed in a circle) or unbounded
- If the feasible region is bounded then both a maximum and a minimum value exist

Graphically Solve

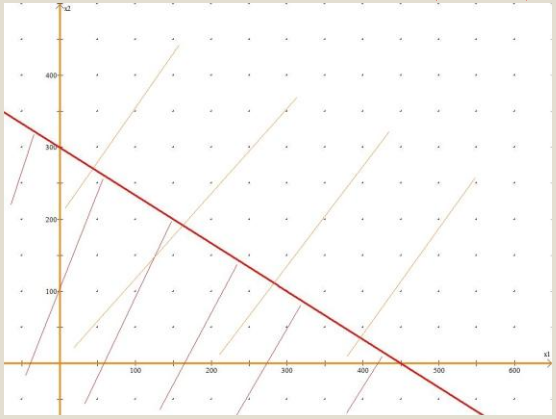


Start by graphing and shading the non-negative constraints

Graphically Solve

$2x_1 + 3x_2 \le 900$

(450,0) , (0,300)

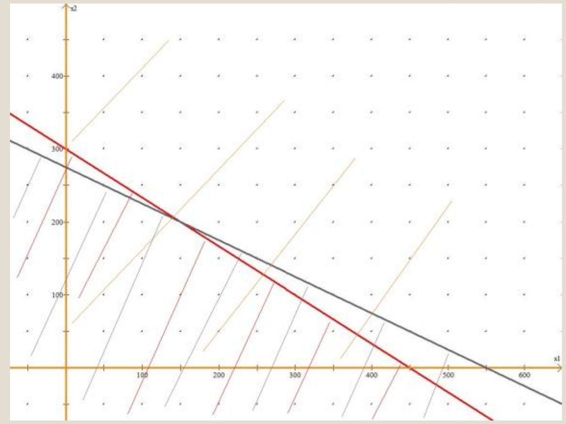


Graph each of the constraints using the x and y intercepts and shade

Graphically Solve

$x_1 + 2x_2 \le 550$

(550,0) , (0,275)

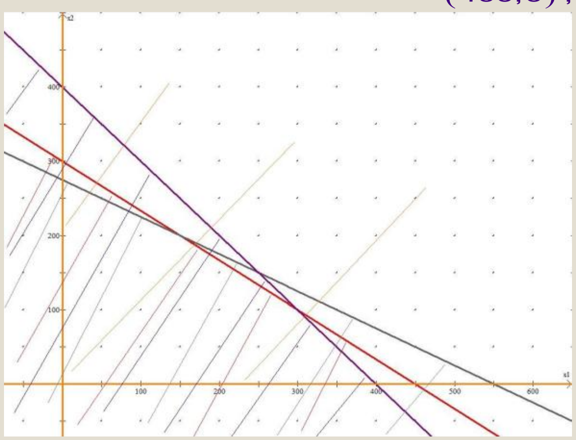


Graph each of the constraints using the x and y intercepts and shade

Graphically Solve

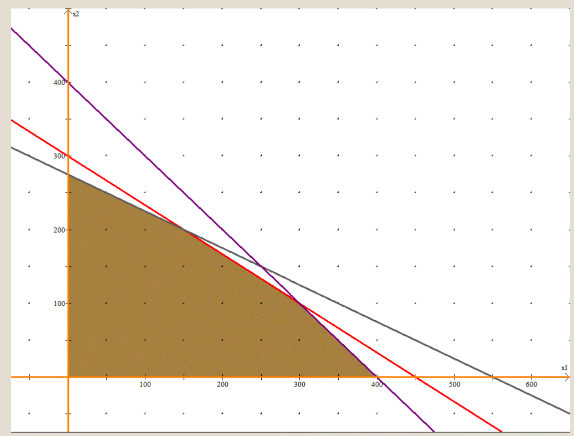
$x_1 + x_2 \le 400$

(400,0) , (0,400)



Graph each of the constraints using the x and y intercepts and shade

Graphically Solve



The feasible region is the intersection of the shaded regions

The corner points are the points of intersection of the boundary lines of the feasible region

Graphically Solve

The feasible region is the intersection of the shaded regions

The corner points are the points of intersection of the boundary lines of the feasible region

Graphically Solve

- The Fundamental Theorem of Linear programming states; if the optimal value exists then that value must occur at one or more of the corner points of the feasible region
- There are two additional possibilities for the existence of an optimal solution
- First, if there is no feasible region the optimal value does not exist
- Second, if the feasible region is unbounded there is no maximum value but the minimum value exists or vice versa but this depends on the coefficients of the objective function

Graphical Solution

Corner Points	$P = 25x_1 + 40x_2$
(0, 0)	0
(0, 275)	11,000
(150, 200)	11,750
(300, 100)	11,500
(450, 0)	11,250

Therefore, the maximum profit is $11,750 when 150 essential covers and 200 essential deluxe covers are produced

Linear Programming

Solving the Problem Method 2 – The Simplex Method

Simplex Method

- The Simplex Method as mentioned previously was developed by George Dantzig in 1947
- The simplex method is an algorithmic process (algebraic) of solving a linear programming problem
- The simplex method is used on a standard linear programming problem in standard form
- The fundamental theorem of linear programming states that if the optimal value exists, then the value must occur at one or more of the basic feasible solutions
- To start the simplex method

Simplex Method

- First you introduce slack variables to change the linear inequalities to equations
- The number of slacks is equal to the number of constraints

$$2x_1 + 3x_2 + s_1 = 900$$
$$x_1 + 2x_2 + s_2 = 550$$
$$x_1 + x_2 + s_3 = 400$$

Simplex Method

- Second you set the objective function to zero maintaining a positive P
- To do this move everything from the right side to the left side in the objective function

$$2x_1 + 3x_2 + s_1 = 900$$
$$x_1 + 2x_2 + s_2 = 550$$
$$x_1 + x_2 + s_3 = 400$$
$$-25x_1 - 40x_2 + P = 0$$

Simplex Method

- Keep in mind the previous slide…
- Form the initial Simplex Tableau

x_1	x_2	s_1	s_2	s_3	P	
2	3	1	0	0	0	900
1	2	0	1	0	0	550
1	1	0	0	1	0	400
-25	-40	0	0	0	1	0

- Notice the negative numbers in the bottom row, this tells us we are not at the maximum value

Simplex Method

- Now we must perform a pivot
- The pivot element is the intersection of the pivot column (the most negative from bottom row) with the pivot row (the smallest quotient of the number on the right divided by the pivot column element)
- A pivot consists of making the pivot element a 1 and everything above and below the 1 a zero
- This process is repeated until there are no negatives in the bottom row
- Very important, you can never pivot on a negative or zero

Simplex Method

- Basic variables – column is a single 1 and zeros and the value is found by "grabbing" the 1 and shooting to the far right
- Non-basic variables – column is random numbers and the value is always zero
- As pivoting is performed a basic variable will become non-basic and vice versa; this is referred to as entering and exiting variables
- Keep in mind… you will never get a negative on the right and your maximum P value will never go down

Simplex Method

$P = 0$ at $x_1 = 0$ $x_2 = 0$

x_1	x_2	s_1	s_2	s_3	P	
2	3	1	0	0	0	900
1	(2)	0	1	0	0	550 ← $\left(\frac{1}{2}\right)$
1	1	0	0	1	0	400
-25	-40	0	0	0	1	0

- So the pivot element is the 2; so make it a 1 and everything above and below the 1 a zero

Simplex Method

x_1	x_2	s_1	s_2	s_3	P		
2	3	1	0	0	0	900	⊕
1/2	(1)	0	1/2	0	0	275	(20) (-3) (-1)
1	1	0	0	1	0	400	⊕
-25	-40	0	0	0	1	0	⊕
-3/2	-3	0	-3/2	0	0	-825	
-1/2	-1	0	-1/2	0	0	-275	
20	40	0	20	0	0	11000	

Simplex Method

$P = 11000$ at $x_1 = 0$ $x_2 = 275$

x_1	x_2	s_1	s_2	s_3	P	
1/2	0	1	-3/2	0	0	75 ← (2)
1/2	1	0	1/2	0	0	275
1/2	0	0	-1/2	1	0	125
-5	0	0	20	0	1	11000

- We are not finished as there is still a negative in the bottom row so find the next pivot element
- So the pivot element is the 1/2

Simplex Method

x_1	x_2	s_1	s_2	s_3	P		
1	0	2	-3	0	0	150	(5) (-1/2)
1/2	1	0	1/2	0	0	275	⊕
1/2	0	0	-1/2	1	0	125	⊕
-5	0	0	20	0	1	11000	⊕
-1/2	0	-1	3/2	0	0	-75	
5	0	10	-15	0	0	750	

Simplex Method

x_1	x_2	s_1	s_2	s_3	P	
1	0	2	-3	0	0	150
0	1	-1	2	0	0	200
0	0	-1	1	1	0	50
0	0	10	5	0	1	11750

- The simplex method is complete
- Therefore, the maximum profit is $11,750 when 150 essential covers and 200 essential deluxe covers are produced

Linear Programming

Today and Future

Today and Future

- Today linear programming is taught in universities, most business schools, industrial engineering departments, and operations research departments, as well as some mathematics departments
- As increasingly more complex problems involving more variables are attempted, the number of necessary operations expanded exponentially and exceeded the computational capacity of even the most powerful computers
- Today most linear programming work is nonlinear based programming, which is beyond the scope of this discussion

Today and Future

- There are numerous areas where operations research has been applied; for example, optimal depreciation strategies; communication network design; computer network design; water resource project selection; bidding models for offshore oil leases; classroom size mix to meet student demand; electric utility fuel management; air-traffic-control simulations; optimal strategies in sports, and many others

Limitations

- In real life situations, when constraints or objective functions are not linear, the simplex method cannot be used
- Factors such as uncertainty, weather conditions, human issues, etc. are not taken into consideration
- The solution may not be an integer, it could be a fraction and the nearest integer may not be the optimal solution
- Only one single objective is dealt with while in real life situations, problems come with multi-objectives

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Questions?

The End
Thank you for coming

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