Renaissance Math

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The medieval period was as barren mathematically as it was in all other areas of intellectual inquiry. Symbolic of this stagnation was the difficulty posed by Euclid's *Elements*. The fifth proposition of the first book proves that the base angles of an isosceles triangle are equal. Medieval students terminated their study of geometry with this theorem because they found its proof too difficult to understand. Around the year 800, the situation began to improve slightly when Charlemagne invited Alcuin of York (735-804) and other scholars to his court and thereby launched what became known as the Carolingian Renaissance. However, it was not until universities in Italy, France, and England began to open starting about 1050 that the groundwork was laid for the widespread rebirth of learning in Europe. The key event was the reintroduction of Aristotle's thought into the West by such figures as Albertus Magnus and especially his student, Thomas Aquinas. Aristotle's vast erudition focused attention on the natural world, and the discovery of works by Euclid, Archimedes, Heron, and others sparked interest in mathematics. Although scholars differ on the initial and final dates of the Renaissance, we will consider the years 1300-1650, starting shortly after the death of Aquinas in 1274 and ending approximately with the death of Galileo and the birth of Newton in 1642.

Let us begin in the late Middle Ages with the best mathematician of that period, Leonardo of Pisa, (ca. 1180-1250), also known as Fibonacci. His interest in mathematics may have been sparked in part by extensive travel representing the business interests of his father, who was a prominent Pisan merchant. In any case, his work helped to prepare the way for the blossoming of mathematics during the Renaissance. At the time, the dominant number systems in use were Roman numerals and the Babylonian sexagesimal system. Leonardo strongly encouraged the adoption of the Hindu-Arabic approach that we employ today. Among his other achievements were the solving of indeterminate equations in the tradition of Diophantus, the use of negative numbers, and the introduction of the horizontal fraction bar notation. Of course, he is best known for introducing the Fibonacci sequence in his famous problem about the reproduction of rabbits. He also contributed to the understanding of irrational numbers by showing that there are irrationals different from those identified by Euclid in the *Elements*. One such number is the only real answer to the equation $x^3+2x^2+10x-20=0$. Fibonacci's rational approximation to this solution was the best estimate to an irrational root ever given to that point in time by a European mathematician.

The development of algebra was one of the major themes of the Renaissance. Among the ancient Greeks, the work of Diophantus in the third century AD was the most algebraic in nature, but it focused on *ad hoc* methods of solving indeterminate equations. The Islamic mathematician, Muhammad ibn Musa Al-Khwarizmi (c. 790-840), founded algebra as a distinct branch of mathematics by discussing general methods applied to types of problems, specifically finding the roots of linear and quadratic equations. In his *Condensed Book on the Calculation of Restoring and Comparing*, he systematically explained how to keep an equation in balance by adding, subtracting, multiplying, or dividing both sides by the same quantity. Al-Khwarizmi's work became so influential that the Arabic word for "restoration", *al-jabr*, led to the formulation of our word, *algebra*.

Nicole Oresme (ca. 1323-1382), a French mathematician associated with the University of Paris, contributed to the advancement of algebra by introducing rational exponents and the rules for computing with them. He also suggested the use of irrational powers although he did not pursue the idea. While studying the concept of uniform acceleration, he decided to represent the relationship between time and velocity graphically, a technique that eventually led to the geometric representation of functions. In the third century B. C., Apollonius had associated curves with equations as part of his monumental work on conic sections. However, he made no attempt to associate equations with curves. By establishing this latter connection, Oresme's work helped to prepare the way for the creation of analytic geometry by Descartes and Fermat in the seventeenth century.

Oresme's interest in change also motivated him to study infinite series. Although the explicit formulation of the concept of a limit lay several centuries in the future, his work stimulated interest in the ideas of convergence and divergence. He gave the first proof that the harmonic series diverges, a demonstration that appears in textbooks to this day.

In the next century another Frenchman, Nicolas Chuquet (1445-1488), introduced negative numbers and zero as exponents. He also dealt with negative coefficients and in some instances negative solutions of equations although he was not consistent in this last respect. His book, *Triparty*, written in 1484, was the earliest work on Renaissance algebra. As the name implies, the book consisted of three parts. The other two dealt with the Hindu-Arabic number system and the calculation of roots.

In 1494, Luca Pacioli (1445-1514) published the first printed work on algebra, entitled the *Summa*, which included material on arithmetic, algebra, Euclidean geometry, and double-entry bookkeeping. Indeed, he is known as the father of double-entry bookkeeping. Other printed algebras appeared in Germany, including one by Johann Widman (ca. 1460), who in 1489 first used the modern symbols for plus and minus, and another by Michael Stifel (ca. 1487-1567), who in 1544 introduced M and D for multiplication and division respectively. A third German mathematician, Adam Riese (1492-1559), popularized computation using the Hindu-Arabic system so effectively that the phrase, *nach* Adam Riese, i. e., according to Adam Riese, is used to this day to attest to the accuracy of a calculation.

Johann Müller (1436-1476) of Königsberg, Germany was one of the most influential mathematicians of the fifteenth century. He adopted the name Regiomontanus, the Latinized version of Königsberg, which translates as King's Mountain in English. Trigonometry as a separate branch of mathematics was inaugurated in the second century B. C. by the Greek mathematician Hipparchus, who developed it primarily to aid his work in astronomy. In the second century A.D., Claudius Ptolemy utilized Hipparchus' work and extended it while writing his *Almagest*, which became the definitive presentation of the geocentric theory of planetary motion and which dominated astronomy until the time of Copernicus. Although there were Latin translations of the *Almagest* at the time, Regiomontanus undertook a new one with an emphasis on the mathematical portions of it. He realized the Ptolemy's discussion of trigonometric results was largely *ad hoc* and that a systematic development of the subject was needed. He established trigonometry as a mathematical discipline independent of astronomy by providing an extensive account of methods for solving triangles and by proving the law of sines.

Regiomontanus also established a printing press at Nuremberg, where he planned to issue translations of Archimedes, Apollonius, Heron, Ptolemy, and others, but his premature death terminated the project. The first mathematical book printed in Europe appeared in 1478 in the town of Treviso, Italy and is known as the *Treviso Arithmetic*. It consisted of stated problems based on commercial applications important to Italian merchants and used the Hindu-Arabic number system. In 1557, Robert Recorde (1510-1558) of Wales published the first English book on algebra. Entitled *The Whetstone of Witte*, it introduced the equality sign that we use today.

In the year, 1545, a seminal event in the history of mathematics occurred when Jerome Cardano (1501-1576), an Italian mathematician and physician, showed how to solve the general cubic and quartic equations by radicals. Since the work of al-Khwarizmi on linear and quadratic equations, no comparable progress had been made on ones of higher degree. Cardano was an unscrupulous character, who originated neither solution. Indeed, he solved the cubic only after receiving a hint from Niccolo Tartaglia, another Italian mathematician. Furthermore, Cardano's secretary, Ludovico Ferrari, was responsible for solving the quartic. Publishing the solution to the cubic equation betrayed Cardano's promise to Tartaglia that he would wait until the latter had done so first. Nevertheless, Cardano was an accomplished mathematician, who had previously learned to rationalize denominators containing cube roots. Complex numbers were poorly understood at the time, and his work stimulated interest in them because square roots of negative numbers appeared in some of the equations he solved. It also motivated mathematicians to attempt to solve the quintic equation by radicals. As we know, this question was resolved negatively in the nineteenth century, but in the process new branches of algebra were created.

Trigonometry received further stimulus from the German mathematician, Georg Joachim Rheticus (1514-1576), a student of Nicolaus Copernicus, who had done extensive work in trig to support his investigations in astronomy. Rheticus published a first account of Copernicus' heliocentric theory of planetary motion, and he was the first mathematician to define the six trigonometric ratios in terms of the sides of a right triangle. He made full use of all six functions because he calculated tables for each of them.

One of the Renaissance's most original contributions to mathematics was the study of perspective, motivated by Italian painters who wanted to develop techniques of representing three-dimensional objects on a two-dimensional surface. The first artist to consider this question seriously was Filippo Brunelleschi (1377-1446), and Leon Battista Alberti (1401-1472) wrote the first book on the subject, in which he commented that the most fundamental requirement of a painter is to know geometry. Leonardo da Vinci (1451-1519) made a similar observation: "Let no one who is not a mathematician read my works." Another artist, Piero della Francesca (1420-1492), advanced Alberti's investigation still further in a book in which he thoroughly discussed the mathematical basis of painting. The interest in perspective spread beyond Italy to Germany because of Albrecht Dürer (1471-1528), who studied for several years in Italy. The field began to receive a formal mathematical treatment with the Frenchman, Girard Desargues (1591-1661), a French engineer and architect, who launched the field of projective geometry.

The concept of projection was important in geography, as well as art. Among ancient scientists, Claudius Ptolemy was the most influential in both geography and astronomy. Just as Copernicus revolutionized astronomy by abandoning the Ptolemaic approach, so Gerard Mercator (1512-1594) broke with it in geography by introducing the projection which is still used to this day and which bears his name.

The leading French mathematician of the sixteenth century was Francois Viète (1540-1603), one of whose innovations was the use of decimal fractions. While Fibonacci had advocated the Hindu-Arabic number system, he had used it for natural numbers only and had continued to express fractional parts in sexagesimal form. Viète completed the transition to a full place-value decimal system.

Viète's most original contributions were to algebra. Although Cardano demonstrated how to solve cubic and quartic equations, he did so by means of numerical examples, not by writing the general equations symbolically. Indeed, while the quadratic formula had been known since the time of the ancient Babylonians, the form in which we write it today had not been devised. Instead of illustrating problem solving techniques by means of specific examples, Viète inaugurated a major advancement in algebraic notation, using vowels to represent unknowns and consonants to represent known values. He was then able to inaugurate the study of the theory of equations by proving relationships among the roots of an equation of a given type rather than focusing merely on the solutions of a specific equation. In the next century, Descartes used letters toward the end of the alphabet for variables and ones toward the beginning for constants, a practice still common today.

Before trigonometry became a separate branch of mathematics, Euclid stated and proved a geometric version of the law of cosines in the *Elements*. Viète formulated the trigonometric version that we use today, and he was also the first mathematician to state the law of tangents. His interest in the analytic aspects of the subject led him to develop the sum to product identities that we use today for sin x + sin y and cos x + cos y, as well as the formulas for sin nx and cos nx.

Another strong advocate of the decimal system was Simon Stevin (1548-1620), a native of Bruges, Belgium. In 1585, he published a pamphlet entitled *The Tenth*, in which he explained in detail how to perform the standard arithmetic processes on fractions written in decimal notation. He also expressed exponents numerically rather than by abbreviations, a step that Viète had not taken. During the 1780s when Thomas Jefferson was the United States ambassador to France, he read and was greatly impressed by Stevin's pamphlet. Upon becoming the first secretary of state in George Washington's administration, he successfully advocated placing our monetary system on a decimal basis.

The ancient Greek mathematicians had distinguished between the concepts of unity and number. For them, unity was the generator of numbers, but the first integer was two, not one. This point of view was accepted until Stevin challenged it by treating the idea of unity as a number. The Greeks had also drawn a sharp distinction between the discrete and the continuous because of the crisis precipitated by their discovery of irrational numbers. Their solution, as formulated by Eudoxus in Euclid's *Elements*, was to treat incommensurability geometrically, not arithmetically. Although a rigorous explanation of irrationals was not developed until the nineteenth century, Stevin viewed them numerically and thereby took an important step toward unifying the number system as we know it today.

The concept of a logarithm was developed in response to the extensive computational needs of astronomy. In 1614, the Scottish mathematician, John Napier (1550-1617), published a logarithmic system that implicitly used 1/e as its base. His work was greeted enthusiastically, and Henry Briggs (1561-1631), an English mathematician, modified it to construct the common log system that we employ today. About the same time, Jobst Bürgi (1552-1632) of Switzerland created a system that was essentially based on e. Natural logarithms as we know them were devised about a century later by Euler. Praising the development of logarithms, Laplace said that "by shortening labors" they "doubled the life of an astronomer."

One of the astronomers who benefited from the use of logarithms was Johannes Kepler (1571-1630). Kepler's fundamental goal was to establish the truth of Copernicus' heliocentric theory by discovering mathematical relationships that govern planetary motion. His efforts culminated in his famous three laws, including the fact that the planets revolve about the sun in elliptical orbits, not circular ones, as Copernicus had assumed. Kepler also succeeded in calculating the area of an ellipse, a result obtained by Archimedes but not known in Europe at the time. The conic sections had first been identified in the fourth century by the mathematician Menaechmus, a member of Plato's Academy. Kepler's use of Greek geometry two millennia later demonstrated its profound impact on Renaissance science.

Galileo (1564-1642) derived a similar benefit from his knowledge of Greek mathematics. Among his vast range of achievements was the discovery that projectiles subject only to the force of gravity travel in parabolic paths. Just as Kepler applied conic sections to describe the path of planets, Galileo used them to explain projectile motion. In doing so, he introduced the idea of a mathematical model to provide a quantitative framework that could be used to make precise predictions about the phenomena he was investigating.

Since their inception, geometry and algebra had developed as separate branches of mathematics. In the first part of the seventeenth century, René Descartes (1596-1650) and Pierre de Fermat (1661-1665), influenced in part by the work of Viète, independently and almost simultaneously integrated the two subjects and created analytic geometry. Kepler's work in astronomy and Galileo's accomplishments in both mechanics and astronomy prepared the way for the emergence of classical physics; analytic geometry provided the basis for the development of calculus, which gave mathematical expression to this revolution in science. The work of Newton and Leibniz was a key component of the next major period in Western history, the Age of Reason.

Raphael's painting, *The School of Athens*, portrays some of the ancient Greek philosophers, mathematicians, and scientists who had an impact on Renaissance thought. For example, two of the figures in the painting are Pythagoras and Claudius Ptolemy. The two men in the center are Plato on the left and Aristotle on the right. Plato is pointing upward as if to his World of Forms. Aristotle, who rejected the existence of the World of Forms, is gesturing toward the earth. As mentioned earlier, it was Aristotle's emphasis on the natural world that sparked the Renaissance. It is interesting to note that Raphael did not entitle the painting, The School at Athens. Although Plato's Academy and Aristotle's Lyceum were two of the most important universities in the ancient world, many of the individuals in the painting attended neither. Rather, Raphael wanted to represent the fundamental connection between the achievements of Greek civilization and the revival of learning in Europe more than a millennium later.

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