Ever wondered if one problem could motivate all of calculus? 
Wonder no more!

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The Problem

Race down a quarter mile drag strip.

Driver’s question: “How many seconds did it take me to reach a speed of 60 mph?”
<table>
<thead>
<tr>
<th>Car #</th>
<th>587</th>
<th>585</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DIAL</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R/T</td>
<td>26.4</td>
<td>73.4</td>
</tr>
<tr>
<td>60’</td>
<td>1.626</td>
<td>2.063</td>
</tr>
<tr>
<td>330</td>
<td>4.673</td>
<td>5.387</td>
</tr>
<tr>
<td>1/8</td>
<td>7.250</td>
<td>8.025</td>
</tr>
<tr>
<td>MPH</td>
<td>94.99</td>
<td>94.23</td>
</tr>
<tr>
<td>1000</td>
<td>9.498</td>
<td>10.312</td>
</tr>
<tr>
<td>E.T.</td>
<td>11.416</td>
<td>12.259</td>
</tr>
<tr>
<td>MPH</td>
<td>117.15</td>
<td>116.14</td>
</tr>
</tbody>
</table>
What would radar tell us?

The driver reaches 60 mph around 4.1 seconds, give or take 0.1 seconds.

Of course, we don’t have to share this information with our students. 😊
Preliminary Analysis

\[ s = -0.2201t^3 + 10.351t^2 + 15.137t - 15.492 \]

\[ R^2 = 1 \]

Time-Distance Graph

Side note:
60 mph = 88 ft/sec
Exploration #1

Can we find two places on the graph where the driver’s **average speed** is near 88 ft/sec?

\[
\frac{s(7.7) - s(0.734)}{7.7 - 0.734} \approx 88.03 \text{ ft/sec}
\]

Even a hasty “by eye” calculation shows \(\Delta s/\Delta t \approx 600/7 \approx 86 \text{ ft/sec}\)
Calculus Ideas

• Mean Value Theorem
• Average Rate of Change
• Instantaneous Rate of Change
• Geometry
  • Secant Lines
  • Tangent Lines
• Local Linearity
Exploration #2

Building a numerical understanding of limit

<table>
<thead>
<tr>
<th>time interval (sec)</th>
<th>average velocity (ft/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[4.04, 7]</td>
<td>108.81</td>
</tr>
<tr>
<td>[4.04, 6]</td>
<td>102.21</td>
</tr>
<tr>
<td>[4.04, 5]</td>
<td>95.17</td>
</tr>
<tr>
<td>[4.04, 4.5]</td>
<td>91.48</td>
</tr>
<tr>
<td>[4.04, 4.3]</td>
<td>89.98</td>
</tr>
<tr>
<td>[4.04, 4.1]</td>
<td>88.46</td>
</tr>
<tr>
<td>[4.04, 4.07]</td>
<td>88.23</td>
</tr>
<tr>
<td>[4.04, 4.06]</td>
<td>88.15</td>
</tr>
<tr>
<td>[4.04, 4.05]</td>
<td>88.07</td>
</tr>
</tbody>
</table>

Instantaneous Velocity = \( \lim_{{t \to 4.04}} (\text{Average Velocity}) = \lim_{{t \to 4.04}} \frac{s(t) - s(4.04)}{t - 4.04} = 88 \text{ ft/sec} \)
Calculus Ideas

• Limit Idea
  • Clustering of values
  • Arbitrarily close

• Definition of the derivative
  • Physical: $s'(t) = v(t)$
  • Geometric: $\frac{\Delta s}{\Delta t} = \text{rate}$

\[
\lim_{t \to 4.04} \frac{s(t) - s(4.04)}{t - 4.04}
\]

foreshadows the more general

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]
Exploration #3

Graph the velocity function, $v(t)$. Since the driver crosses the finish line at 116.14 mph, every speed from 0 mph to 116.14 mph must have been achieved.

In particular, there must have been a time $t = c$ where $v(c) = 60$ mph or 88 ft/sec.
Calculus Ideas

• Continuity

• Intermediate Value Theorem
  • “c” can be found here
  • Corvette context gives authentic meaning to the “c” in the IVT
Exploration #4

\[ \int_{0.734}^{12.259} v(t) \, dt = s(12.259) - s(0.734) \]

\[ \approx 1319.05 \text{ ft} \]

\[ \approx \frac{1}{4} \text{ mile} \]

= length of drag strip

\[ \int_{0.734}^{4.04} v(t) \, dt = s(4.04) \approx 199 \text{ ft} \]

(tells us WHERE on the drag strip the driver reached 88 ft/sec)

(approximate graph)
Calculus Ideas

• \[ \int v(t) \, dt = s(t) \]

• **Fundamental Theorem of Calculus**

\[ \int_{a}^{b} v(t) \, dt = s(b) - s(a) = \text{displacement} \]

\[ \int_{a}^{x} v(t) \, dt = s(x) \quad \text{with} \quad x = 4.04 \text{ sec--a precursor initial} \]

to the accumulation function \( A(x) = \int_{a}^{x} f(t) \, dt \).
Exploration #5

What was the driver’s **average speed** over the quarter mile?

\[
v_{avg} = \frac{1}{12.259 - 0.734} \int_{0.734}^{12.259} v(t) \, dt \\
= \frac{s(12.259) - s(0.734)}{12.259 - 0.734} \\
\approx 114.45 \text{ ft/sec} \\
\approx 78 \text{ mph}
\]
Calculus Ideas

• Average value of $f$ on $[a,b]$

$$f_{avg} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx$$
References


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