Motivating Students to Learn Infinite Series

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Non-Trivial Problems

For which values of b does the following converge?

$$\sum_{n=1}^{\infty} b^{\ln n}$$

Rainfall Example

In a millennium, how many years will have more rainfall than any year preceding it, in other words, how many years would we expect to have the rainfall record be broken?

Keith Nabb's Question

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if the following conditions are met.

(1) $\lim_{n \to \infty} a_n = 0$

(2) $a_{n+1} \leq a_n$ for all $n \geq N$

Question: Is it possible for a series to converge if the second condition is not met?

Mathematics Teacher 104(5) 373-378

Question

Why should students learn about infinite series?

Philosophical Question

Can a person add up an infinite collection of positive numbers and get a sum that is a real number?

Not-so Philosophical Question

If the sequence of terms has a limit of zero, must the infinite series converge?

How Can We Motivate Students?

How Can We Motivate Students?

- Historical Perspectives
- Connect Series to Prior Learning
- Applications within Mathematics
- Applications from Other Disciplines
- Visual/Tabular Representations
- Non-Trivial Problems

Zeno's Paradox

- Achilles and the Tortoise
- *In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.* – as recounted by Aristotle, *Physics* VI:9, 239b15

Zeno’s Paradox

- Super Model Version

Historical Perspectives

Oresme ca 1350

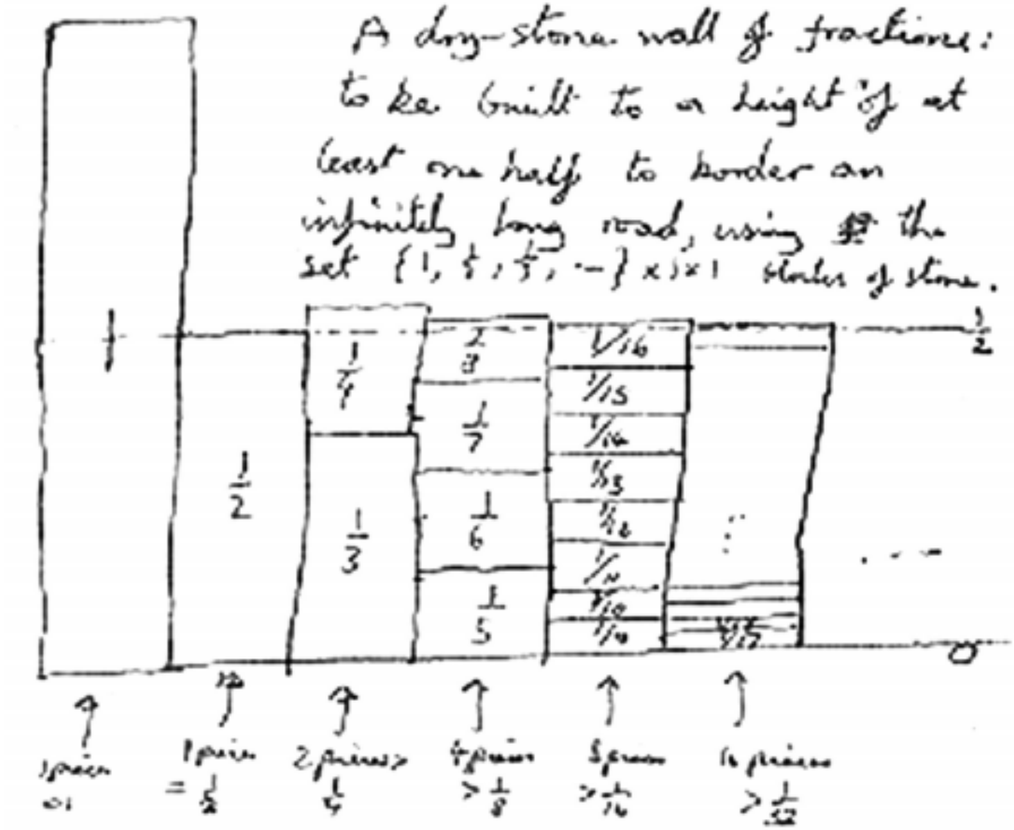
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\cdots$$

$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+$$

$$\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)+\cdots$$

Historical Perspectives

Oresme ca 1350



Gonzalez-Martin (2011)

Historical Perspectives

- Aristotle - Physics, III, IV, 206b, 1-33.
- Archimedes - Quadratura parabolae
- Andreas Tacquet (1612-1660)
- Guido Grandi (1671-1742): 1-1+1-1+… = 1/2
- Leibniz (1646-1716) argued using probability to support Grandi's claim.
- Riccati (1676-1754) argued against Grandi
- Gauss (1777-1855) defined convergence "correctly."

G. Bagni: University of Udine

Connect Series to Prior Learning

Topics from Pre-calculus

- Converting Repeated Decimals to Fractions.
- Identifying Convergent and Divergent Geometric Series.

True or False

$.99999\ldots < 1$

Paradox?

$$1+2+4+8+\cdots=S$$

$$2+4+8+\cdots=2S$$

Paradox?

$$1 + 2 + 4 + 8 + \cdots = S$$

$$2 + 4 + 8 + \cdots = 2S$$

$$\left(1 + 2 + 4 + 8 + \cdots\right) = S = 2S - S$$

Paradox?

$$1+2+4+8+\cdots=S$$

$$2+4+8+\cdots=2S$$

$$\left(1+2+4+8+\cdots\right)=S=2S-S$$

$$2S-S=\left(2+4+8+\cdots\right)-\left(1+2+4+8+\cdots\right)=-1$$

Decimals

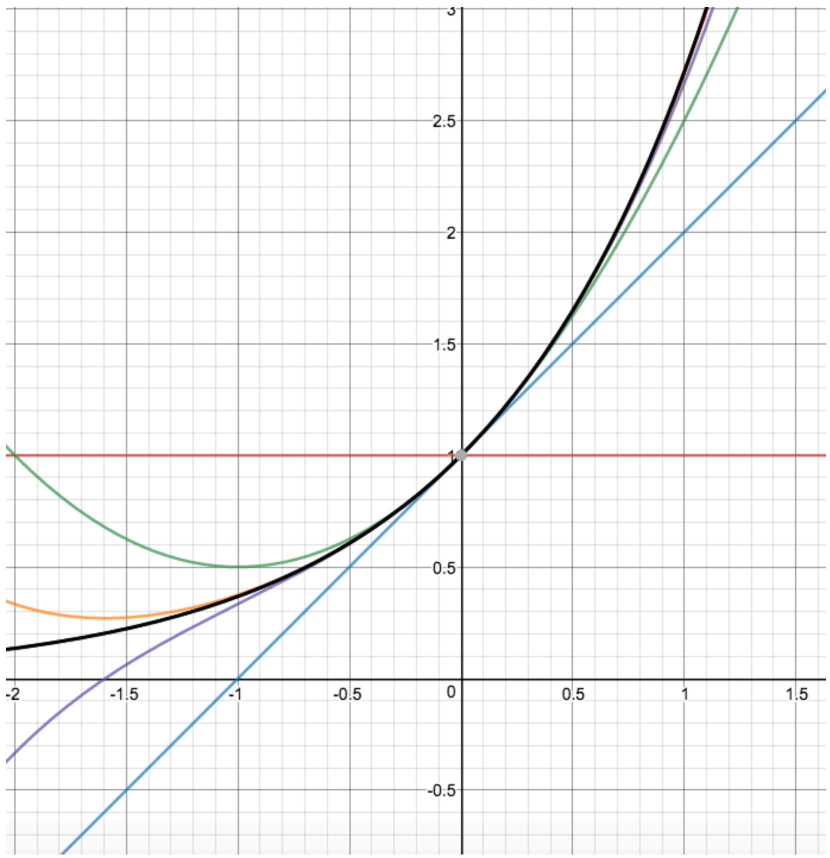
- What is a non-terminating non-repeating decimal?
- What is a terminating or repearing decimal decimal?
- How are these concepts related to series?

Applications within Mathematics

Taylor Series

- Limits, Derivatives and Intgrals are much easier using Taylor Series.
- Quantify the amount of error in an approximation scheme like Newton-Raphson or Simpson's Rule.
- Find approximate solutions to many types of differential equations.

Emerging Functions



$y = e^x$ approximations with Taylor Polynomials

Interesting Facts

Euler's Rule

$$e^{ix} = \cos x + i \sin x$$

$$e^{i\pi} + 1 = 0$$

Applications Outside Mathematics

Economics Example

If the government gives you a $100 tax rebate, and if you (and others) spend $90 and save $10, what will be the overall net increase in GDP?

http://math.stackexchange.com/questions/9524/motivating-infinite-series

Economics Example

- Suppose the government gives you a $100 tax rebate. You will save, say, $10, and spend the other $90. So at first glance, it seems that the $100 rebate increased GDP by $90.

Economics Example

- But the $90 you spent will not just disappear; it will go to other people, who will in turn save 10% of it, and spend the other 90%. The 90% the second round spent will go to a third round of people who spend 90% of that, which goes to a fourth round of people...

Economics Example

$$90 + 81 + 72.9 + \cdots$$

$$= \sum_{i=1}^{\infty} 100(0.9)^i$$

$$= \frac{90}{1 - 0.1} = 900$$

Economics Example

How much should you invest today at an annual interest rate of 5% compounded continuously so that you can make annual withdrawals of $2000 in perpetuity?

Owen K. Davis

Mediaspace.itap.purdue.edu

Economics Example

$$2000 = P_1 e^{(0.05)(1)}$$

$$2000 = P_2 e^{(0.05)(2)}$$

$$2000 = P_3 e^{(0.05)(3)}$$

Economics Example

$$2000 = P_1 e^{(0.05)(1)}$$

$$2000 = P_2 e^{(0.05)(2)}$$

$$2000 = P_3 e^{(0.05)(3)}$$

$$P_1 = \frac{2000}{e^{(0.05)(1)}}; \quad P_2 = \frac{2000}{e^{(0.05)(2)}}; \quad P_3 = \frac{2000}{e^{(0.05)(3)}}; \cdots$$

$$P = P_1 + P_2 + P_3 + \cdots$$

Economics Example

$$P_1 = \frac{2000}{e^{(0.05)(1)}}; \quad P_2 = \frac{2000}{e^{(0.05)(2)}}; \quad P_3 = \frac{2000}{e^{(0.05)(3)}}; \cdots$$

$$P = P_1 + P_2 + P_3 + \cdots$$

$$P = \sum_{n=1}^{\infty} \frac{2000}{e^{0.05n}} = \frac{2000/e^{0.05}}{1 - e^{-0.05}} = \$39,008.33$$

Economics Example

$$P = \sum_{n=1}^{\infty} \frac{2000}{e^{0.05n}} = \frac{2000/e^{0.05}}{1-e^{-0.05}} = \$39{,}008.33$$

But what if the interest rate is 1% instead?

Economics Example

$$P = \sum_{n=1}^{\infty} \frac{2000}{e^{0.05n}} = \frac{2000/e^{0.05}}{1-e^{-0.05}} = \$39{,}008.33$$

But what if the interest rate is 1% instead?

$$P = \sum_{n=1}^{\infty} \frac{2000}{e^{0.01n}} = \frac{2000/e^{0.01}}{1-e^{-0.01}} = \$199{,}001.67$$

Rainfall Example

- In a millennium, how years will have more rainfall than any year preceding it, in other words, how many years would we expect to have the rainfall record be broken?

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In a millennium, how many years will have more rainfall than any year preceding it, in other words, how many years would we expect to have the rainfall record be broken?

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{1000}\approx 7.485$$

Rainfall Example

If we extend the time from 1000 years to an infinite number of years, how many years of record rainfall can we expect?

https://plus.maths.org/content/perfect-harmony

Physics

In special relativity, where the Maclaurin series are used to approximate the Lorentz factor, taking the first two terms of the series gives a very good approximation for low speeds. You can actually show that at low speeds, special relativity reduces to classical (Newtonian) physics.

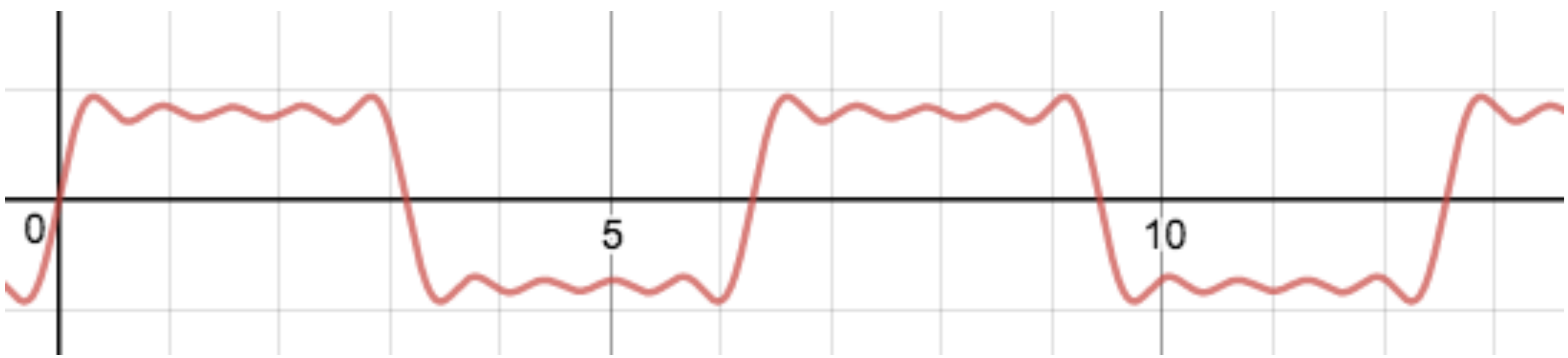
http://math.stackexchange.com/questions/218421/what-are-the-practical-applications-of-the-taylor-series

Gibbs Phenomenon

In acoustic engineering it is important to model square wave functions.

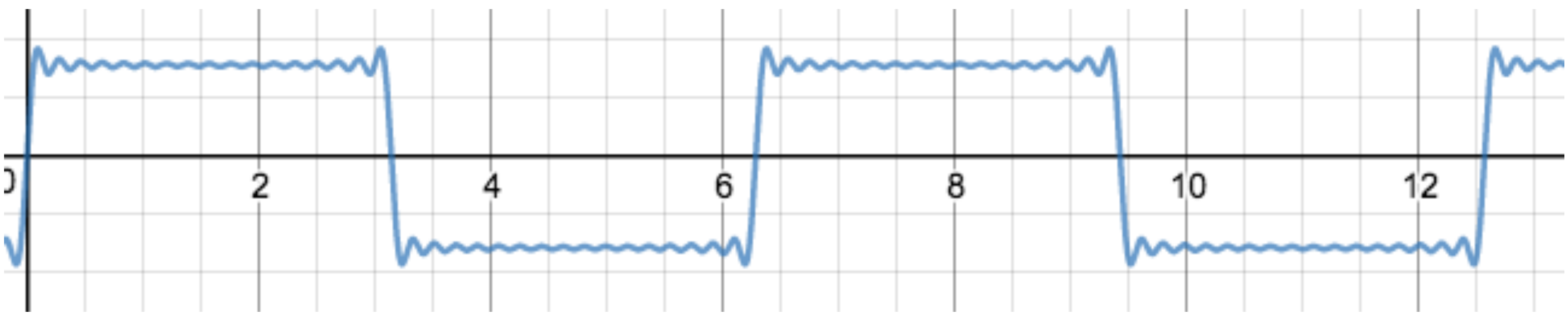
Gibbs Phenomenon

$$y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \frac{1}{9}\sin 9x$$



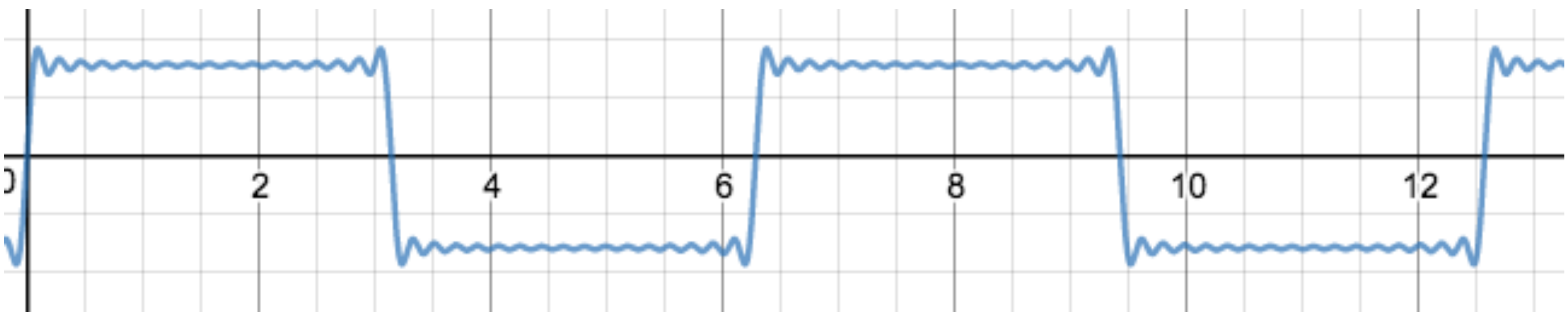
Gibbs Phenomenon

$$y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots + \frac{1}{33}\sin 33x$$



Gibbs Phenomenon

$$y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots + \frac{1}{33}\sin 33x$$



The overshoot is always about 9% at every discontinuity,

Fourier sums overshoot at a jump discontinuity, and that this overshoot does not die out as more terms are added to the sum.

Gibbs Phenomenon

Fourier sums overshoot at a jump discontinuity, and that this overshoot does not die out as more terms are added to the sum.

https://en.wikipedia.org/wiki/Gibbs_phenomenon

Gibbs Phenomenon

In the graphics world these kinds of "ringing" effects are things to be eliminated if possible. For example some kinds of compression artifact are a form of the Gibbs phenomenon. But with audio, if you want to play a piecewise linear signal say, you have to do the opposite. You need to find all of your discontinuities and add in a suitable ringing effect.

https://plus.google.com/+DanPiponi/posts/ej2D8RxJkLg

Compounded Simple Interest

You invest $1000 in an account paying 5% simple interest. You earn interest on the interest only after it accumulates to $1000.

Shapiro, Isidor F. "Harmonic Series in Interest Problems." National Mathematics Magazine (1939): 230-230.

Thanks to Steve Kifowit for passing this on.

Compounded Simple Interest

You invest $1000 in an account paying 5% simple interest. You earn interest on the interest only after it accumulates to $1000.

For the first twenty years, the account earns $50 per year. After 20 years the total interest is $1000 and it is eligible to earn interest.

Compounded Simple Interest

For years 21-30, the account earns $100 per year. Another $1000 of interest is generated and it is eligible to earn interest.

Compounded Simple Interest

For years 21-30, the account earns $100 per year. Another $1000 of interest is generated and it is eligible to earn interest.

Now there is $3000 earning interest. It will take one-third of ten years or three and one-third years before there is enough interest to begin earning additional interest.

Compounded Simple Interest

U_1 becomes U_2 $(=U_1+1U_1)$ in $1\,p$;

U_2 next becomes $U_3(={}^3/_2\,U_2=U_2+{}^1/_2U_2)$ in ${}^1/_2\,p$;

U_3 next becomes $U_4(={}^4/_3\,U_3=U_3+{}^1/_3U_3)$ in ${}^1/_3\,p$;

. . . . .

and we next attain U_n.............................in $\dfrac{1}{n-1}\,p$.

Compounded Simple Interest

U_1 becomes U_2 $(=U_1+1U_1)$	in 1 p;
U_2 next becomes $U_3(={}^3/_2\,U_2=U_2+{}^1/_2U_2)$	in ${}^1/_2$ p;
U_3 next becomes $U_4(={}^4/_3\,U_3=U_3+{}^1/_3U_3)$	in ${}^1/_3$ p;
.....	
and we next attain U_n..............................	in $\dfrac{1}{n-1}\,p$.

If we represent by y_n the time for attaining to U_n, then it is obvious that

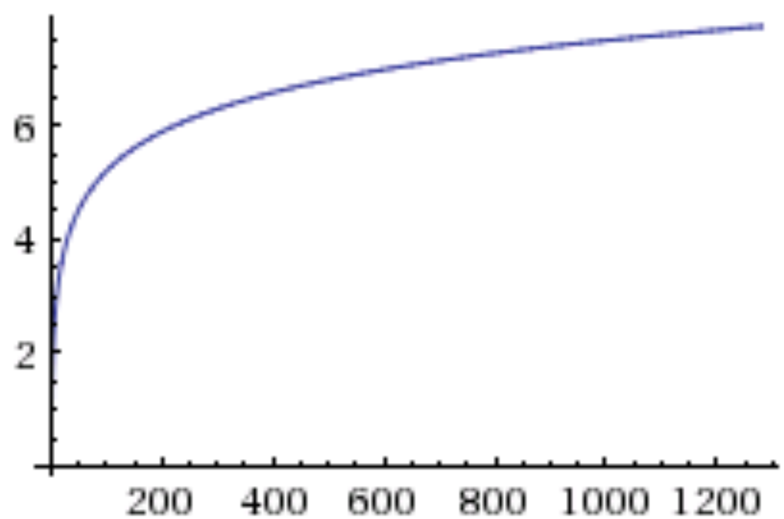
$$y_n=p\left\{1+{}^1/_2+{}^1/_3+{}^1/_4+\cdots\cdots+\frac{1}{n-1}\right\}=p\cdot H_{n-1}.$$

Visual/Tabular Representations

Visual/Tabular Representations

$$\sum_{n=2}^{\infty} \frac{1}{n}$$

Partial sums:



Visual/Tabular Representations

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

n	Sn
100	2.0424
1000	2.0554
10000	2.06000
100000	2.06211
1000000	2.06327
10000000	2.06396

Non-Trivial Problems

Non-Trivial Problems

It is known that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Use this fact to prove that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots=\frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\cdots=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{24}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots=\frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\cdots=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{24}$$

$$1+\frac{1}{9}+\frac{1}{25}+\cdots=\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8}$$

Non-Trivial Problems

For which values of b does the following converge?

$$\sum_{n=1}^{\infty} b^{\ln n}$$

$$z = b^{\ln n}$$

$$\ln z = \ln b^{\ln n}$$

$$\ln z = (\ln n)(\ln b)$$

$$z = b^{\ln n}$$

$$\ln z = \ln b^{\ln n}$$

$$\ln z = (\ln n)(\ln b)$$

$$z = e^{(\ln n)(\ln n)}$$

$$z = n^{\ln b}$$

$$z = \frac{1}{n^{-\ln b}}$$

$$z = b^{\ln n}$$

$$\ln z = \ln b^{\ln n}$$

$$\ln z = (\ln n)(\ln b)$$

$$z = e^{(\ln n)(\ln n)}$$

$$z = n^{\ln b}$$

$$z = \frac{1}{n^{-\ln b}}$$

$$-\ln b > 1$$

$$\ln b < -1$$

$$b < e^{-1}$$

$$b < \frac{1}{e}$$

Keith Nabb's Question

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if the following conditions are met.

(1) $\lim_{n\to\infty} a_n = 0$

(2) $a_{n+1} \leq a_n$ for all $n \geq N$

Question: Is it possible for a series to converge if the second condition is not met?

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Nabb's Student Responses

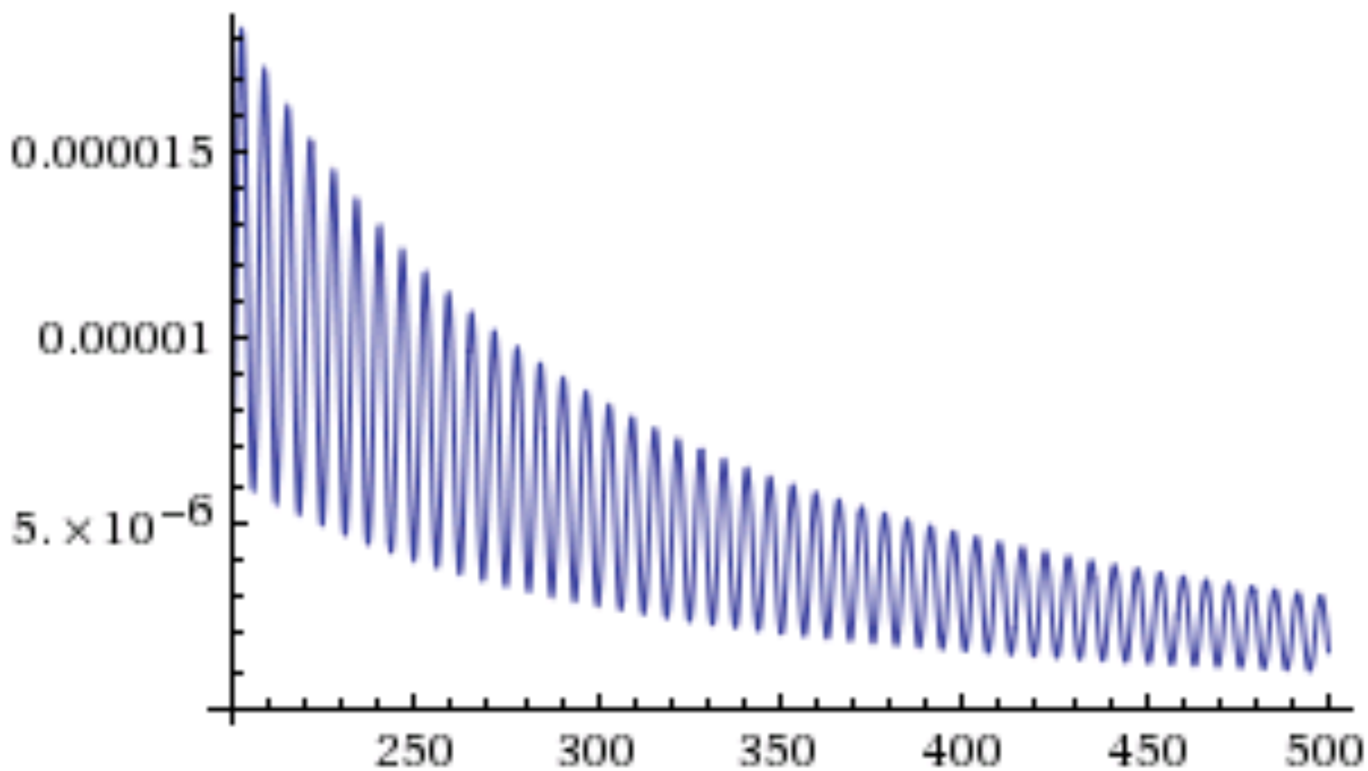
Laura:

$$\sum_{n=1}^{\infty}(-1)^2\left(\frac{\sin n+2}{4n^2}\right)$$

$\left(\frac{\sin n+2}{4n^2}\right)$ for $n=1..20$

0.7103677462, 0.1818310892, 0.05947555578, 0.01942496101,
0.01041075725, 0.01194850349, 0.01355605407, 0.01167718065,
0.007444810139, 0.003639947223, 0.002066135937, 0.002540672017,
0.003580128753, 0.003814550198, 0.002944764267, 0.001671969417,
0.0008984450760, 0.0009637444088, 0.001488834632, 0.001820590782

Nabb's Student Response



Nabb's Student Response

Tom

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{9}+\frac{1}{8}-\frac{1}{27}+\cdots.$$

$$a_n=\begin{cases}1/2^{(n+1)/2}, & n=1,3,5,7,\ldots \\ 1/3^{n/2}, & n=2,4,6,8,\ldots\end{cases}.$$

Questions or Comments

Thank You!

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See you in San Diego.