Looking Ahead: Teaching Calculus in the 22nd Century

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Calculus Today

Calculus today often constitutes a substantial body of material to teach, especially for students planning to major in math, certain sciences and engineering. There are certain elementary techniques of differentiation and integration that are important. When these are combined in various ways it can provide problems that are challenging. Thus, this aspect of calculus is not simply a matter of drill and following reductionist rules. The reductions can be quite challenging. Also, in the evolution of calculus, we have seen one of the most significant attempts in mathematics to encompass various notions of infinitesimal and infinity to rules of calculation. This is essentially an accommodation of the notion of infinity to finite algorithms.

Calculus techniques are a focus because these are basic to applications of calculus in science and engineering. In addition, there are basic intuitions about these techniques, and aspects of formal rigorous mathematics that are worthwhile to introduce to a certain extent. Calculus can often provide bridges between rigorous mathematics and intuitions that can be interesting, and helpful in learning the meaning of some of the concepts and techniques. The elements of rigor were developed through basic ideas in analysis and topology that were investigated in the nineteenth century to avoid paradox and nonsensical results in the uncritical use of calculus. This is especially important for mathematics majors, but can also be important for others, like physics majors.

Teaching Calculus

To a certain extent, teaching calculus amounts to developing in the student skill at playing a certain formal game, as Hilbert considered math, on the whole to constitute. At the core of the formal results, there are several basic theorems, the principal one being the fundamental theorem of calculus. In thinking about and discussing these theorems, the student can often improve in the use of calculus, as well as come to appreciate some of the formal aspects of the subject. It is through the acquisition of this “language game”, to borrow a term from Wittgenstein, that high levels of proficiency in calculus and the imaginative application of calculus can be achieved. Wittgenstein’s notion of game was, however, deeper than Hilbert’s and includes the teacher transmitting a certain “culture” and various points of view to the student. This “game-playing” leads directly into mathematical modeling, which is a critical, but difficult, aspect of calculus to teach.

A teacher must take into account the relationship of calculus to prior education of the student in algebra. An important side of calculus is the enhancements calculus provides for making progress in school in knowledge of algebra, trigonometry and analytic geometry. Some puzzling aspects of these areas, too, can be explained using calculus much more easily than using other approaches. A student who studies calculus diligently often improves their knowledge of these other areas very significantly. Also, the study of these other areas leads to introductions to some fundamentally revealing and important results in abstract mathematics, such as Stokes' theorem.

Calculus Uses

Related to mathematical modeling are the numerical techniques, some of which, at least, a teacher usually includes in basic calculus. There are numerous applications related to physics and numerical analysis, but which can be more broadly applied, such as Simpson's rule for integration. The study of these applications is a significant introduction to applied mathematics and the study of differential equations. Differential equations can be taught to a limited extent in a calculus course. There are broader concerns, such as linear algebra. The applied mathematics from calculus and linear algebra can significantly reduce the difficulty involved in learning theoretical physics in the introductory levels of physics and beyond. In fact, the main application of calculus is to physics and related fields of engineering, and this not only relates to physics majors but to engineering and science undergraduates, because much theoretical knowledge in these areas at this level has been structured by calculus and linear algebra.

Assumptions

The teacher must often rely on the student having acquired a substantial background in algebra. It usually requires a fairly good knowledge of algebra and trigonometry to study calculus in its full extent for engineering and science. Much computational skill needs to be acquired by the student as a calculus course progresses. In addition, the principles of calculus can involve more rigor in mathematical development than what the student has been accustomed to see in preceding courses. Accompanying the study with analytic geometry, including both single-variable and multivariable calculus, and treating the theorems and definitions with adequate attention both to rigor and the development of intuition, can take as much as two years of coursework, including elementary differential equations, linear algebra, and advanced calculus. The full extent of applications of calculus is impressive, but especially so in classical physics, with classical mechanics and electromagnetism. Calculus is not only important for various fields of physics, engineering, technology and other areas of applied science, but was culturally and historically significant in the evolution of civilization. Overall, we are considering a very rich area in mathematics, both broadly significant and deep.

Teaching of calculus has been strongly influenced by the industrial revolution. The industrial revolution gradually led to the idea of the skilled specialist, and therefore a need for universal and mass education. This is the background, for the most part, for current calculus courses. The demands of industry, due to a lot of inherent complexity in design and manufacture, seem to require specialization. People can learn specialties through a sequence of interconnected courses, designed for fairly general purposes, which when combined for a major, would lead to a competent level of skill for entry into the specialist area. The interconnectedness is partially revealed in that the requirements of one course could be fulfilled by taking certain prior courses. The student selects a specialization, and the teacher teaches certain courses for which they have been well-prepared by prior certification. Calculus, in this light, and the development of the associated points of view related to computation, is a centrally important suite of courses.

Mass Education

Overall, this system of universal and mass education, actually mirroring industry itself, along factory, assembly-line methods, has worked satisfactorily in virtually every type of situation in which modern industrial techniques have dominated in the world. This has proven to be independent of the mode of economic system in place, whether communist or capitalist, and of other factors that we might think of as cogent. The teachers for calculus have usually been fairly well-prepared in the subject matter through their own specialized training, and often naturally-talented at teaching. It has almost universally been possible to design majors for a great multitude of specializations requiring standardized calculus courses.

For calculus, this has meant that there are several types of calculus courses. There is a calculus course that focuses on drill, intuition and rules for the business majors. There is a calculus focusing on drill, intuition and rules for the engineering majors, but which concerns more of the basic mathematical structure needed for applied mathematics in engineering, and some emphasis on the viewpoints of mathematics of actual mathematicians. This course, or rather sequence of courses, is generally much more demanding than the course for business majors. Finally, there is a calculus course that focuses on perceived needs of math majors for rigorous and structured development of mathematical ideas. This course bears a significant resemblance to the sequence of courses for the engineers, but is usually more rigorous, and usually requires at least one more course in advanced calculus, where significant theory is developed and rigor is important. Teachers are not usually equally-well suited to teach each of these sequences. In fact, it is common for mathematicians who have specialized in an area of modern mathematics to teach advanced calculus, because of its close relationships with real and complex analysis, and to differential geometry. A significant difference between teachers with a more applied bent and such pure mathematicians is often that pure mathematicians teach the course sequence for math majors while applied mathematicians often teach the course sequences for business and engineering majors. While there is little to distinguish a “pure” mathematician from an “applied” mathematician at the highest levels of mathematics, important differences in points of view can arise at the level of teaching calculus. This is especially the case with respect to mathematical modeling, where the perspectives of the pure and applied mathematician are expected to be very different.

Limitations of the Current System

We, as teachers, are all familiar with some of the limitations of the current system of education in calculus. For example, there is a clear mismatch between introductory engineering physics and engineering calculus in terms of timing. Many students are not introduced to critical elements of calculus until the third semester of calculus that are used in the first semester of engineering physics. There are especially egregious mismatches in terms of postponing the discussion of several variables to the third semester of calculus, when knowledge of multivariable calculus and linear algebra are definitely advantageous for the first semester of physics. Also, theorems such as Stokes' theorem are useful in the first and second semester of engineering physics, while such geometric theorems are often not discussed, if discussed at all, until the third semester of calculus. Students cannot often have taken the third semester of calculus prior or in coordination with the first semester of physics.

Another common problem is that "word problems", usually training for mathematical modeling, are often the most important types of problems in calculus for applied courses such as physics. On the other hand, teachers of calculus often focus on abstract problem-solving, on drill and on reductionist techniques, and slight word problems. The result is that, for example, students of introductory physics will encounter problems exactly those they solved as word problems in calculus, but due to the abstraction of the material in calculus, fail to recognize this important fact.

Teachers often confront a lack of intellectual maturity in classrooms. We mention that students take calculus at points in their lives where they are often unable to see its importance or acquire points of view conducive to learning calculus. A lack of intellectual maturity and flexibility in points of view and perspectives, are problems not only for students learning calculus but for learning mathematics at many levels. This is often only a first indication, in calculus, of troubles the student is going to have accommodating the general, abstract level in modern science and engineering. There is almost an unavoidable level of abstraction in academics that can hinder people in pursuing practical careers in business, science and engineering. For many students this becomes seriously noticeable in calculus courses, and can pose ongoing difficulties beyond calculus in the courses for their majors or specializations. We may think that calculus "shouldn't" be so abstract and theoretical, but the trend in academics has if anything made the teaching of calculus more and more abstract, as students get introduced to the full complexity entailed in their specializations and academic majors.

Questions about the Traditional Approach to Education

The industrial model for mass education has persisted and evolved for over two hundred years. We need to emphasize that despite serious limitations and flaws, the assembly line system has functioned satisfactorily. Teachers, for the most part, are able to adapt to limitations. This traditional system works satisfactorily, in different ways, of course, for all countries where industrialization and mass-production have been important, and where specialist experts were needed.

The adequacy of this traditional approach to calculus is coming into question, as computer technology advances and our civilization is becoming more and more affected by the computer and modern technology. All of us from the older generations recognize how much information is now accessible because of this computer revolution. More and more we are seeing, due to this availability, efforts being made to “equalize the playing field”, so that education can become less standardized and more personalized. The elite, with resources and privileged access to traditional education, and therefore much easier routes to specialization than most other people, are now no longer seen as privileged. We have tried to equalize opportunities and privileges. People are attempting to use the internet and other computer technologies to circumvent the traditional assembly line. There has been mixed success at this, and it has gotten entangled with all sorts of political maneuvering. Nevertheless, it seems clear that education in the future is going to be more personalized. What implications does this have for the teaching of calculus? Will there even be such standardized courses in calculus in the 22nd century such as we see today? Clearly, the computer revolutionize must be regarded as distinct from the industrial revolution, and its consequences are going to affect the traditional ways that grew up around the industrial revolution.

Personalization and Fragmentation

We see the educational system as very much in flux with respect to such changes. In the long run, however, a much higher level of personalized training than exists today seems inevitable. It seems apparent that for each specialization needed in an organization, we will first see to what extent that specialization can be accommodated by robotics and automation. What remains, which will often be of a rather simple nature, but nevertheless not accessible currently to robotics and automation, will need to be done by a human. These simple tasks will not require the type of complex specializations that exist today. The whole vast apparatus of assembly line education, for complex science or engineering tasks will become obsolete. Instead, a need will develop to demonstrate simple skills for the specialist to carry out specific and practical tasks. For this type of education, we will need predominately laboratory-style courses, where each student is involved in lab work pertaining to their specialization. Such a high level of personalized instruction can be carried out by the company or other organization that the person works for, or can be the task of specialized laboratory facilities.

The Evolving Pragmatic Age

To what extent can theoretical, innovative, creative and abstract pursuits be accommodated in such an extremely practical system, at least as we see culture evolving in the U.S.? To a certain extent, not. We envision an essential “dark age” relative to many common pursuits today that are of a more abstract nature, not tied to the direct practical needs of an organization. In short, the standardized calculus courses of today will very likely be gone. They will instead, be replaced by specialized laboratory facilities designed to satisfy certain practical needs. There will still be much innovation, but it will be more of a practical nature, intended in part to develop robotics and automation to eliminate the need for a specialist altogether. If a student must use differentiation, there will be a focus on exactly what practical skill will be needed. The student may need to recognize when the chain rule needs to be applied: automation does the rest. This is the implication of personalization: the student will get practical training directed toward their specific job at the moment. There will be no need to take a whole sequence of calculus courses, and no need to discuss rigorous or abstract aspects of mathematics.

The Teacher as Technician

Can we say anything more definite about teaching calculus in this far future, but perhaps not so far as well? As education becomes more personalized, the teacher becomes more of a technician. This is someone who supplies and maintains the technical learning laboratories students will need. A skill with computer interfaces with the student, and with sensors and control of mechanical devices used in developing student abilities will become essential. Transmitting technical knowledge of how to work with hardware and software will become an important skill for a teacher.

This evolutionary trend in teaching is a result of a combination of complex factors, leading to ever more highly specialized technical staff. A result of this, together with personalized education directed toward highly specialized needs of organizations, is the fragmentation of knowledge. Only aspects of calculus critical to job performance need be taught. This is possible because the computer revolution makes information highly accessible, and math software mitigates the level of proficiency in calculus that a person need attain. Indeed, there need be no general level, and the whole system of higher education, as we know it today, will be modified greatly by this fragmentation of subject matter and personalization of education.

Algorithmic Approach to Mathematics

In addition, mathematics itself, apart from the artistry of pure mathematics, is going to become more and more absorbed into algorithmic approaches, and the need for methods, such as linear algebra and discrete mathematics, dominating much of a need for basic calculus instruction. With a growing focus on laboratory work in mathematics, our culture will inevitably tie mathematics to very specific and applied methods of statistical approaches and linear, nonlinear and discrete analysis. We expect the focus of teaching in calculus directed more toward modeling, using linear analysis and differential equations. The basic, abstract and more foundational material, which is less useful for applications and for laboratory work in areas related to applied mathematics is likely to see much neglect, except in the community of pure mathematicians, who will presumably pursue mathematics for its artistry and gamesmanship.

The Twenty-Second Century

The future of teaching calculus, into the twenty-second century, will depend on a complex of factors. Certainly, some relatively minor current trends may come to dominate the teaching of calculus. The current trends toward technical expertise in various specialized ways, and the personalization of education tend to suggest, overall, a fragmentation of calculus tailored to the needs of particular organizations and individuals, and away from the current “factory” style of education today. What we view as teaching calculus today seems to be headed toward fragmentation into various technical specialties. The teaching of calculus, as a result, is likely to improve markedly through this enhanced specialization. However, fragmentation of the nature of teaching is likely to come at a cost.

Can we say something specifically directed toward calculus teaching? The student in the 22nd century is likely to rely much less on technical skill and expertise, as this will very likely be accommodated in entire packaging directed toward their specific job and subjective needs, through applications of artificial intelligence, automation and robotics, with directed software and hardware. Let us suppose, for example, that a student is in need skill of with a second-order differential equation with constant coefficients. The role of the teacher will be to guide the student through appropriate experiments, software and use of technology. The direct use of techniques of calculus will be unnecessary, except in unusual instances, and in those instances, a technical expert would be brought into help, perhaps not a human but an artificial intelligence. Calculus will be integrated into the devices the student and teacher use. Learning to work with calculus will be similar to learning to drive a car, with the teacher similar to driving instructors. Furthermore, it will be unnecessary, except in the case of certain highly specialized individuals, who may not be human, to actually acquire technical expertise with calculus. Calculus will always just be an integrated part of a system that the student must learn to “drive”.

We will see, with the fragmentation of education and learning geared toward personalization, much more effective learning. This will be accompanied with integrated technology, with less need for calculus courses, or other technical courses as we now understand them. The teacher will have a distinctly and very different role in such a world than in our current culture. We can see this much about the distant future. Such a world is likely to seem almost alien and unrecognizable to us living today. Although we cannot make specific predictions, these seem to be at least trends.