Medieval Mathematics

The medieval period in Europe, which spanned the centuries from about 400 to almost 1400, was largely an intellectually barren age, but there was significant scholarly activity elsewhere in the world. We would like to examine the contributions of five civilizations to mathematics during this time, four of which are China, India, Arabia, and the Byzantine Empire. Beginning about the year 800 and especially in the thirteenth and fourteenth centuries, the fifth, Western Europe, also made advances that helped to prepare the way for the mathematics of the future.

Let us start with China, which began with the Shang dynasty in approximately 1,600 B. C. Archaeological evidence indicates that long before the medieval period, the Chinese had the idea of a positional decimal number system, including symbols for the digits one through nine. Eventually a dot may have been used to represent the absence of a value, but only during the twelfth century A. D. was the system completed by introducing a symbol for zero and treating it as a number. Other features of the Shang period included the use of decimal fractions, a hint of the binary number system, and the oldest known example of a magic square.

The most significant book in ancient Chinese mathematical history is entitled *The Nine Chapters on the Mathematical Art*. It represents the contributions of numerous authors across several centuries and was originally compiled as a single work about 300 B. C. at the same time that Euclid was writing the *Elements.* However, in 213 B. C., a new emperor ordered the burning of all books written prior to his assumption of power eight years earlier. It is not certain, however, to what extent the decree was executed, and part of the *Nine Chapters* was reconstructed from fragments recovered after the burning. The version of it that survives today was prepared in A. D. 263 by the mathematician, Liu Hui (3rd century), and consists of a commentary on and extension of the original work. It had a fundamental impact on Chinese mathematics during the medieval period.

The *Nine Chapters* contains 246 arithmetic, geometric, and algebraic problems and their solutions. It is similar to the *Elements i*n that it represents China's accumulated mathematical knowledge until the middle of the third century A. D. However, it differs fundamentally from Euclid's work because it lacks a deductive structure. Results are typically stated but not proved. For example, to calculate square roots, Chinese mathematicians used the algorithm based on the squaring of a binomial that some of us were taught in school. By applying a similar procedure derived from the formula for cubing a binomial, they found cube roots. In neither case was a theoretical justification for the correctness of the method given.

In geometry, China focused on the development of formulas for the calculation of the areas of circles, rectangles, triangles, and trapezoids, and the volumes of spheres, cylinders, pyramids, and cones. Some of their results were exact; others were only approximate. For example, four ways to calculate the area of a circle were given: three were correct while the fourth assumed that π is equal to three.

The Chinese were aware of the Pythagorean theorem, and Liu Hui referred to a diagram in the *Nine Chapters* to explain it. His discussion did not constitute a proof, but the result was used extensively to solve problems involving right triangles. Geometry was also applied to questions of surveying and employed techniques that we would justify by reference to similar and congruent triangles.

Chinese mathematicians were fascinated by the number π and sought to approximate it ever more accurately. Archimedes (287-212 B. C.) had used the method of exhaustion to trap its value between 3 10/71 and 3 1/7. Although Liu Hui was unaware of Archimedes' work, he used the same method to arrive at the improved approximation of 3.14159.

Two hundred years later, Tsu Chung-Chi (430-501) and his son located π between 3.1415926 and 3.1415927. In addition to his work in mathematics, Tsu was an excellent astronomer. Today there is a crater on the moon named for him.

In algebra, the *Nine Chapters* solved a system of simultaneous linear equations by writing it in matrix form and using a method equivalent to Gaussian elimination. As we would anticipate, negative numbers sometimes appeared in the process of implementing this technique. In fact, the Chinese were the first civilization to recognize the existence of negative numbers, and they understood the rules for computing with them.

Achievements of the thirteenth century represented the zenith of Chinese mathematics in the medieval period. Qin Jinshao (1202-1261) generalized the process of finding square and cube roots mentioned earlier to give a systematic account of solving polynomial equations in his work, *Mathematics in Nine Sections* (not to be confused with the *Nine Chapters*). We know this approach as Horner's method because the English mathematician, William Horner, essentially rediscovered it in 1819. It consists of estimating a solution of an equation, making a substitution based on the estimate, and solving the resulting equation.

The last significant Chinese mathematician of the thirteenth century was Chu Shih-Chieh (1249-1314), who wrote a book entitled *Precious Mirror of the Four Elements.* It began with a diagram of what we call Pascal's triangle and gave binomial expansions through the eighth power. In addition to discussing Horner's method, he stated correct formulas for the sum of the first n positive integers, the sum of their squares, and the sum of the first n triangular numbers. As was customary in Chinese mathematics, no proofs of the results were included.

Beginning in the fourteenth century, Chinese mathematics went into decline. About 250 years later, European missionaries introduced the country to Euclid's *Elements* with its axiomatic foundation and logical structure. These features, which, as we have seen, had not been utilized in China, gradually began to replace the traditional approach.

Let us now turn to India. Although Hindu civilization began by at least 2,000 B. C., there is no evidence of significant mathematical activity for more than a millennium. About 800 B. C., a decimal number system with individual symbols for the digits from one through nine began to develop although it did not include zero and positional notation was not used.

There are only a few records of early Hindu mathematical advances prior to the medieval period. In 499, Aryabhata (476-550) wrote the first connected account of the subject, a work entitled *Aryabhatiya.* It included techniques for the calculation of square and cube roots, the use of 3.1416 as an approximation of π, and accurate formulas for the areas of rectangles, circles, and trapezoids. However, its results for the volume of a pyramid and a sphere were incorrect.

Aryabhata hinted at the use of a positional decimal number system although he did not discuss the idea explicitly. As we mentioned, the Hindus had employed ten as a base for centuries, but they had also introduced individual symbols for some numbers larger than nine. Around the year 600, they abandoned those additional symbols and adopted the place value approach, including the concept of zero as a number, not merely as a placeholder. In doing so, they created the first positional decimal system in history. We saw that the Chinese adopted the idea of place value at an earlier date, but they did not include zero until the twelfth century.

Aryabhata also discussed a major Hindu contribution to trigonometry. It is difficult to ascertain the degree of contact between the Chinese and other cultures, but the Hindus became aware of developments in Greek mathematics following Alexander the Great's conquest of part of India. In particular, they learned of the creation of trigonometry, which was founded by Hipparchus in the second century B. C. The Hindus took Hipparchus' definition of the sine function based on chords of a circle and modified it to create the one we use today. They also implicitly worked with the cosine function.

Brahmagupta (598-670), one of the leading Hindu mathematicians of the seventh century, provided extensive discussions of the use of the positional decimal number system in arithmetic calculations. He formulated the standard rules for operating with rational numbers, and he recognized the existence of negative numbers. We have seen that the Chinese encountered negative numbers while solving systems of equations. Brahmagupta introduced them as a means of representing debts, and like the Chinese, he stated the correct rules for combining them. The Greeks had formulated these principles geometrically in terms of subtraction; Brahmagupta converted them to their arithmetic equivalents although he did not understand that division by zero is undefined. In algebra, Brahgamupta recognized that quadratic equations have two solutions, which he found by completing the square. He accepted irrational as well as negative answers although he rejected square roots of negative numbers.

Another major Hindu mathematician of the seventh century was Bhaskara I (600-680), so designated in order to distinguish him from a later one with the same name. He wrote on the positional decimal number system and is credited with introducing the use of a circle to represent zero. In his trigonometric work he investigated connections between the sine and cosine functions, and he used the concept of a reference angle to discuss the sine values of angles in the second, third, and fourth quadrants.

For many years Hindu mathematics was motivated primarily by questions in astronomy. In the ninth century however, Mahavira wrote the first textbook that treated it as an independent subject. One of his interests was combinatorics. To calculate the number of combinations of n objects taken r at a time, he devised an approach equivalent to the one we use today.

The leading Hindu mathematician of the twelfth century was Bhaskara II (1114-1185), who wrote on a wide variety of topics, including linear and quadratic equations, arithmetic and geometric progressions, irrational numbers, and Pythagorean triples. Despite the fact that Hindu mathematics did not emphasize geometry and seldom proved its results, he used similar triangles to give a rigorous proof of the Pythagorean theorem. His most famous work is entitled *Lilavati*.

In their study of trigonometry, the Hindus originally constructed a table for the sines of angles between zero and ninety degrees in increments of 3.75 degrees. To find the sines of other angles, they used linear interpolation. With the passage of time, areas such as navigation required more accurate values. In response, the mathematician Madhava (1359-1425) foreshadowed aspects of calculus by developing equivalents of the Maclaurin series for the sine and cosine functions.

As in China, mathematics in India went into decline, but Arabic mathematicians were among those who benefited from Hindu achievements. In the century between 650 and 750, the Arabs gained control of a vast swath of territory stretching from parts of Spain to India. The museum and the library in the city of Alexandria had been the mathematical center of the Hellenistic era. A comparable institution, called the House of Wisdom, was established at Baghdad during the medieval period, and scholars were invited to join it. One of these was Mohammed ibn-Musa al-Khwarizmi (780-850), who wrote two books of fundamental importance to the history of mathematics. The first one, probably based on a translation of the work of Brahmagupta, gave a systematic presentation of the Hindu number system and was so complete that today we call it the Hindu-Arabic number system.

Al-Khwarizmi's second book was of even more importance because it established algebra as a distinct branch of mathematics by focusing on general methods of solving linear and quadratic equations. In particular, like Brahmagupta, he solved quadratics by completing the square, and he also developed rules for various algebraic processes, such as the multiplication of two binomials. Diophantus' solution of Diophantine equations was the most algebraic of the Greek mathematical accomplishments, and he is sometimes referred to as the father of algebra. However, the title belongs more appropriately to al-Khwarizmi. Diophantus' work was restricted to solving specific problems by *ad hoc* techniques. Today it is considered to be part of number theory rather than algebra.

Al-Khwarizmi's work added two new words to the vocabulary of mathematics. The term "algorithm" was derived from his name and referred to computations made with the Hindu-Arabic system. Our word "algebra" comes from the Arabic *al-jabr* , which appeared in the name of his second book and which means "restoration." It apparently referred to the rebalancing of an equation by transferring a term from one side to the other.

In addition to creating a new branch of mathematics, Arab mathematicians preserved critical Greek manuscripts. Thabit Ibn-Qurra (826-901) translated the works of Euclid, Archimedes, Apollonius, and Ptolemy into Arabic and included commentaries on them. Without his effort, books five, six, and seven of Apollonius' *Conics* would have been lost. Among his other contributions, he provided additional proofs of the Pythagorean theorem and a discussion of magic squares.

We have seen that the Hindus based trigonometry on the sine function rather than the Greek use of chords. Although the Arabs were aware of both approaches, they adopted the Hindu alternative and extended it by introducing the tangent and cotangent functions. The secant and cosecant ratios were also used, although less frequently.

Omar Khayyam (1050-1123) is familiar to us as the Persian poet who wrote the *Rubaiyat*. However, he also made a significant contribution to mathematics by finding solutions of cubic equations. Like al-Khwarizmi, he solved quadratics both algebraically and geometrically, but he believed that cubics could be solved only geometrically. His approach was to make a substitution in an equation that allowed him to find its solutions as the points of intersection of two conic sections. He restricted himself to positive answers only because, unlike the Chinese and the Hindus, Arabic mathematicians did not recognize negative solutions. It was not until the sixteenth century in Renaissance Italy that an algebraic method of solving cubic equations was developed.

The ancient city of Byzantium was founded about 657 B. C. by the Greeks. Centuries later, it fell under Roman control, and in 330, its site became the location of the new city of Constantinople, founded and named by the emperor Constantine I in response to threats of external invasion and internal intrigue. His action effectively divided the Roman Empire into western and eastern parts with Rome and Costantinople as their respective capitals. The Eastern Roman, or Byzantine Empire, as it came to be called, survived the fall of Rome in 476 and remained autonomous for almost a millennium. Although weaker mathematically than the Chinese, Hindu, and Arabic civilizations, it still played a vital role in medieval intellectual history. After the Byzantine emperor Justinian closed Plato's Academy in 529, Constantinople was one of the places where scholars relocated.

The key contribution of the Byzantine Empire to mathematics was the preservation of Greek manuscripts, the most spectacular example of which was a compilation of Archimedes' works by Isidore of Miletus (442-537) about 530. Isidore was one of the last directors of the Academy before he came to Constantinople. A copy of his collection was made in the tenth century, and three hundred years later, a second document was written over it to create what is known as the Archimedes Palimpsest. It contained the only known versions of two Archimedean texts, *Stomachion* and *The Method of Mechanical Problems*. In the former, Archimedes became the first mathematician to introduce the topic of combinatorics; in the latter, he explained how his analysis of problems in physics helped him to use the method of exhaustion to calculate areas, volumes, and centers of gravity, an approach that foreshadowed the development of integral calculus. The palimpsest was stored unnoticed in a library in Constantinople until it was discovered in the nineteenth century. When archivists examined it, they also found the only Greek-language copy of *On Floating Bodies*, the treatise in which Archimedes established the study of hydrostatics.

Byzantine mathematics also emphasized the writing of commentaries on the works of earlier authors. Nicomachus of Gerasa (60-120), a town located in modern Jordan, studied at Alexandria, where he wrote his *Introduction to Arithmetic*. The book was quite elementary, but it revitalized Pythagorean number theory by discussing such topics as perfect and triangular numbers and various proportions. It was used as a text for centuries, in part because it was written from the perspective of the philosophy of Neopythagoreanism, which followed the Pythagorean tradition of attaching occult properties to numbers, a common practice in the Byzantine Empire. As one indication of the book's enduring popularity, two commentaries by Byzantine authors were written on it, one in the sixth century and the other five hundred years later.

With the collapse of Rome in the fifth century, Western Europe entered the Dark Ages, during which time the vast intellectual achievements of the Hellenic and Hellenistic periods of Greek history were lost. Symbolic of this stagnation was the difficulty posed by Euclid's *Elements*. The fifth proposition of the first book proves that the base angles of an isosceles triangle are equal. Medieval students terminated their study of geometry with this theorem because they found its proof too difficult to understand.

Anicius Boethius (480-524) was one of the few mathematical links between Rome and the past. He understood the low level to which education had sunk, and he wrote texts on each of the subjects recognized at that time as mathematical: arithmetic, geometry, astronomy, and music, known collectively as the quadrivium, a term Boethius himself introduced. Each of these books, however, was quite elementary. His *Arithmetic* was a summary of Nicomachus' *Introduction to Arithmetic*, and his *Geometry* contained statements without proof of some of the propositions in the first four books of Euclid's *Elements*. His work on astronomy was based on Ptolemy's *Almagest*, and the one on music drew from Euclid, Nicomachus, and Ptolemy.

Boethius belonged to a distinguished Roman family. He became a major adviser to Theodoric, one of the kings who ruled Italy after the collapse of Rome. However, he was unjustly accused of conspiring to overthrow the government and was executed after a long incarceration. While in prison, he wrote his most famous work, *The Consolation of Philosophy*.

The ancient world divided areas of scholarly inquiry into two groups: the mathematical and the nonmathematical. The first was the quadrivium, mentioned above; the second, eventually known as the trivium, consisted of logic, grammar, and rhetoric. As the seven liberal arts, they constituted the curriculum of formal education for centuries. Magnus Aurelius Cassiodorus (480-575), one of Boethius' contemporaries, wrote explanations of each of the seven, which, although quite simple, were used in the church schools of the time.

Throughout the seventh century the quality of education continued to decline. Among the few contributions to mathematics, Isidore of Seville (570-636) (not to be confused with Isidore of Miletus) wrote a short summary of Boethius' *Arithmetic*. The Englishman Bede (673-735), known as the Venerable Bede, focused on the mathematics required for the ecclesiastical calendar, especially establishing the date of Easter.

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The Dark Ages ended, and the Middle Ages began with Charlemagne (742-814) about the year 800. Although illiterate himself, he recognized the need for education, and he launched what is known as the Carolingian Renaissance by inviting scholars, including the Englishman, Alcuin of York (ca. 735-804), to come to his court. The most accomplished intellectual of his time, Alcuin introduced an improved curriculum based on the liberal arts. He wrote texts for each of the seven areas, searched for additional manuscripts to be read by his students, and is thought to have composed a book entitled *Propositions for Sharpening Youthful Minds*, which consisted of fifty-three logical and arithmetic puzzles.

Educational progress was slow in Charlemagne's empire, which started to disintegrate shortly after his death because of barbarian invasions. In the tenth century, Gerbert (ca. 940-1003) was born in France and eventually became Pope Sylvester II, the first French pope. He developed an interest in mathematics and wrote on arithmetic and geometry. His most significant mathematical achievement was the introduction of the Hindu-Arabic number system into Europe although he may not have understood that zero can be treated as a number.

During the twelfth century the translation into Latin of the philosophical, mathematical, and scientific works preserved by the Arabic and Byzantine civilizations was crucial to the intellectual revival of Western Europe. Gerard of Cremona (fl. 1150-1185) led a team that translated some ninety volumes, including Euclid's *Elements*, Archimedes' *Measurement of a Circle*, Apollonius' *Conic Sections*, and Ptolemy's *Almagest*. The translation of al-Khwarismi's *Algebra* in 1145 by the Englishman, Robert of Chester, marked the beginning of the study of algebra in Europe. Another Englishman, Adelard of Bath (fl. 1116-1142), issued editions of the *Elements* and the arithmetic contributions of al-Khwarizmi.

The best mathematician of the late Middle Ages was Leonardo of Pisa (ca. 1180-1250), also known as Fibonacci. His interest in mathematics may have been sparked in part by extensive travel representing the business interests of his father, who was a prominent Pisan merchant. He strongly encouraged the adoption of the Hindu-Arabic system, including the use of zero as a number. Among his other achievements were the solving of indeterminate equations in the tradition of Diophantus, the introduction of negative numbers into Europe, and the adoption of the horizontal fraction bar notation inherited from the Arabs. Of course, he is best known for the Fibonacci sequence, 1, 1, 2, 3, 5, 8,…, which appears in his famous problem about the reproduction of rabbits. He also contributed to the understanding of irrational numbers by showing that there are irrationals different from those identified by Euclid in the *Elements*.

Nicole Oresme (ca. 1323-1382), another key figure during the late medieval period, was born in Normandy and attended the University of Paris. He advanced algebra by introducing rational exponents and the rules for computing with them; he also considered the possibility of irrational exponents although he lacked the notation necessary to pursue the idea. While studying the concept of uniform acceleration, he decided to display the relationship between time and velocity graphically, a technique that led to the geometric representation of functions and eventually contributed to the creation of analytic geometry.

In order to calculate the distance traveled by an object subject to non-uniform velocity, Oresme investigated infinite series. Although the explicit formulation of the concept of a limit lay several centuries in the future, his work stimulated interest in the ideas of convergence and divergence. He gave the first proof that the harmonic series diverges, a demonstration that appears in calculus textbooks to this day.

Let us conclude by examining the key event that ended the Middle Ages and launched the Renaissance. Toward the end of the twelfth century, universities began to open, the oldest of which were at Bologna, Paris, and Oxford. The curriculum was based on the quadrivium and the trivium; of fundamental importance were the works of Aristotle on logic and science, all of which had been translated into Latin. Augustine (354-430), the most influential philosopher of the early part of the medieval period, was Platonic in his outlook and had discouraged the investigation of nature in favor of otherworldly concerns, but Aristotle refocused attention on the natural world.

The Jewish philosopher, Moses Maimonides (1135-1204), and the Arabic scholar, Averroes (1126-1198), both of whom were born in Córdoba, Spain, disseminated Aristotle's views. Each of them influenced the German scholar, Albert the Great (1200-1280), who wrote commentaries on virtually all of the Aristotelian treatises and who was the teacher of Thomas Aquinas (1224-1275). The foremost Aristotelian philosopher of his time, Aquinas taught at the University of Paris and made Aristotle's thought acceptable to the medieval world. Within a century of his death, the rebirth of the pursuit of knowledge known as the Renaissance had begun.

The painting, *The School of Athens*, by the Renaissance artist Raphael encapsulates this last point superbly. At the center are Plato pointing toward his world of forms and Aristotle gesturing toward the earth. Not all of the other figures have been identified with certainty, but in the lower left hand corner we have Pythagoras, Boethius, Averroes, and perhaps Nichomachus. On the other side are Ptolemy and either Euclid or Archimedes. It is significant that Raphael entitled the painting, *The School of Athens*, not *The School at Athens*. Many of the individuals he included never worked in Athens, but they contributed to the scholarly life of their time. He captured the profound contribution of the Greeks to civilization, the preservation and continuation of their work elsewhere in the world during the medieval period, and the resurgence of their intellectual interests in Europe with the advent of the Renaissance.

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