A Talk on Real Infinite Series

Pre-Algebra through Calculus II

1st Grade

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Infinite sequence: \( \{a_n\} = a_1, a_2, a_3, \ldots \)
0.3, 0.03, 0.003, 0.0003, 0.00003, ... converges to 0

Infinite series: \( \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \)
0.3 + 0.03 + 0.003 + 0.0003 + ... converges to 1/3

Convergence tests for series

- \( r \)-test
- Comparison tests
- Integral test
- Geometric series
- Telescoping series
- p-series
- Ratio & root tests

Purpose of talk:

- Explore fun real infinite series examples and proofs.
- Provide ideas for introducing infinite series to students who are not yet in college level mathematics.

Zeno of Elea (ca. 490-430 BC)

Zeno’s Dichotomy Paradox:

“That which is in locomotion must arrive at the half-way stage before it arrives at the goal.”

Aristotle, ca. 350 BC

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 1
\]

\[
\sum_{n=1}^{\infty} \tan^{-1} \left( \frac{2}{n^2} \right) = \frac{3\pi}{4}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{4}
\]

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2
\]

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty
\]
Don Cohen "The Mathman" 1930-2015
- Calculus By and For Young People (ages 7, yes 7 and up)
- Website: mathman.biz

Partial sums notation:
\[ S_1 = \frac{1}{2} \]
\[ S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \]
\[ S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \]
\[ S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} \]

The sequence of partial sums approaches 1, and therefore, 
\[ \sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1. \]

A geometric series with first term \( a \) and common ratio \( r \) converges to 
\[ \frac{a}{1-r} \] if and only if \(-1 < r < 1\).

Proof: 
\[ S_n = a + ar + ar^2 + \cdots + ar^{n-1} \]
\[ rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \]

\[ S_n - rS_n = a - ar^n \]
\[ S_n = \frac{a - ar^n}{1-r} \]

As \( n \to \infty \), \( ar^n \to 0 \) if and only if \(-1 < r < 1\).

\[ \sum (\text{Geometric Series}) = \frac{\text{first term}}{1-\text{common ratio}} \] for \(|r| < 1\).

The Harmonic Series: 
\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \]

- For compilations of proofs of the divergence of the harmonic series, see references at the end of this talk.
- Perhaps the most commonly shown proof is Oresme’s proof and the integral test proof (neither of which I include in this talk).
- My goal was to find a proof which I could show pre-algebra students.

Let’s assume that the harmonic series converges to a finite sum \( S \), and group the terms in pairs:

\[ S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \]
\[ = (1 + \frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6}) + (\frac{1}{7} + \frac{1}{8}) + \cdots \]
\[ > (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{2} + \frac{1}{2}) + \cdots \]
\[ = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = S \]

\( S > S \) is a false statement, and so by contradiction, the harmonic series cannot converge to \( S \), and therefore must diverge.

- Leonard Gillman (1917-2009), from "Leonard Gillman: An Interview (Part 2)" by Rachel Metzke
Or, we could form groups of three terms...

\[ S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \frac{1}{13} + \frac{1}{21} + \cdots \]

\[ = \left(1 + \frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{5} + \frac{1}{8} + \frac{1}{13}\right) + \left(\frac{1}{21} + \frac{1}{34} + \cdots\right) \]

\[ > \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{8} + \frac{1}{13}\right) + \left(\frac{1}{21} + \frac{1}{34} + \cdots\right) \]

\[ = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \]

\[ = S \]

... or groups of \( k \geq 2 \) terms

\[ S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \frac{1}{13} + \frac{1}{21} + \cdots \]

\[ = \left(1 + \cdots + \frac{1}{k}\right) + \left(\frac{1}{2k+1} + \cdots + \frac{1}{3k}\right) + \cdots \]

\[ > \left(\frac{1}{2k+1} + \cdots + \frac{1}{3k}\right) + \cdots \]

\[ = \frac{k}{k} + \frac{k}{2k} + \frac{k}{3k} + \cdots \]

\[ = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \]

\[ = S \]

Grouping (starting with \( \frac{1}{2} \)) according to the Fibonacci Sequence 1, 1, 2, 3, 5, 8, ...

\[ \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \frac{1}{13} + \frac{1}{21} + \cdots \]

\[ = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{8}\right) + \left(\frac{1}{13} + \frac{1}{21} + \frac{1}{34}\right) + \cdots \]

\[ > 1 + \frac{1}{2} + \frac{1}{3} + \frac{2}{5} + \frac{3}{8} + \frac{5}{13} + \frac{8}{21} + \cdots \]

\[ = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \frac{1}{13} + \frac{1}{21} + \cdots \]

But \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges by the Test for Divergence because

\[ \lim_{k \to \infty} \frac{F_{n+1}}{F_n} = \lim_{k \to \infty} \frac{F_{n+1}}{F_n} = \lim_{k \to \infty} \left(1 + \frac{1}{F_n}\right) = 1 - \frac{1}{\varphi} \neq 0 \]

(The proof that \( \lim_{n \to \infty} \left(\frac{F_n}{F_{n+1}}\right) = \phi = \frac{\sqrt{5} + 1}{2} \) is left as an exercise.)

We were allowed to group terms in the previous proofs, without worrying about finding a formula for the \( n \)th partial sum, because the terms were all positive.

Remember: If terms are not all positive, watch out!

\[ -1 + 1 - 1 + 1 - 1 + \cdots = (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots \]

\[ = 0 + 0 + 0 + \cdots \]

\[ = 0 \]

\[ -1 + 1 - 1 + 1 - 1 + \cdots = -1 + (1 - 1) + (1 - 1) + (1 - 1) + \cdots \]

\[ = -1 + 0 + 0 + 0 + \cdots \]

\[ = -1 \]

\[ -1 + 1 - 1 + 1 - 1 + \cdots = 1 - 1 + 1 - 1 + \cdots \]

\[ = 1 + 0 + 0 + 0 + \cdots \]

\[ = 1 \]

Back to the harmonic series!

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \]

Here are a few of my favorite proofs that I don't typically show my students...

The Bernoulli Brothers Johann and Jacob

- Johann's proof was published by his brother Jacob in 1689.
- Jacob's proof, also from 1689, is not as well known as Johann's.
- Johann's proof is sometimes erroneously referred to as Jacob's proof. Or, it is sometimes referred to as "the Bernoulli proof."
Johann's proof:
First, note that \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \) is telescoping and converges to 1. Form the following sums:

\[
\begin{align*}
C &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots = \frac{1}{2} \\
D &= \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots = \frac{1}{6} \\
E &= \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots = \frac{1}{12} \\
F &= \frac{1}{20} + \frac{1}{30} + \cdots = \frac{1}{20} \\
G &= \frac{1}{30} + \cdots = \frac{1}{30} \\
\end{align*}
\]

Since the series \( C + D + E + F + G + \cdots \) has only positive terms, it must either converge, or diverge to infinity. Assume that it converges.

Adding the first and second columns:

\[
C + D + E + F + G + \cdots = 1 + \frac{1}{2} + \frac{3}{6} + \frac{4}{20} + \frac{5}{30} + \cdots = 1 + 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots
\]

The above contradiction shows that \( C + D + E + F + G + \cdots \), and therefore the harmonic series, must diverge.

Jacob's proof:
First, let \( j \) be a positive integer. Form the sums

\[
\sum_{n=1}^{j} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j},
\]

\[
\sum_{n=j+1}^{\infty} \frac{1}{n} = \frac{1}{j+1} + \frac{1}{j+2} + \cdots
\]

Note that both sums have \( j \) terms, and that the second sum equals \( (j^2 - j) \left( \frac{1}{j^2} \right) = 1 - \frac{1}{j} \).

Note also that

\[
\sum_{n=j+1}^{\infty} \frac{1}{n} > \sum_{n=j+1}^{\infty} \frac{1}{j^2}
\]

Adding \( j \) to both sides of the above inequality, we have:

\[
\sum_{n=1}^{j} \frac{1}{n} > 1 - \frac{1}{j}
\]

Now let's define the sequence \( a_i = 2, a_{i+1} = (a_i)^2 + 1 \).

(Continued)

Alternating Harmonic Series

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln(2)
\]

- Converges by alternating series test
- Variety of proofs for showing its sum is \( \ln(2) \)

\[
f(x) = \frac{1}{x}
\]

Area = 1 - \( ( \text{area of white rectangle} ) = 1 - \frac{1}{2} + \frac{1}{3} \)
A formal proof that the alternating harmonic converges to $\ln(2)$

Step 1: Let $y_n = H_n - \ln(n + 1)$, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

Prove that $\{y_n\}$ converges.

Step 2: Note that $\lim_{n \to \infty} (H_n - \ln(n))$ is the same as $\lim_{n \to \infty} (H_n - \ln(n + 1))$. Call this number $\gamma$.

Step 3: Prove that $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \ln(2)$

Step 1: Let $y_n = H_n - \ln(n + 1)$, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

Prove that $\{y_n\}$ converges.

Proof:

From the picture, we see that

$H_n > \int_{1}^{n+1} \frac{1}{x} dx = \ln(n + 1)$

And therefore, $y_n = H_n - \ln(n + 1) > 0$

Also from the picture, note that

$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \int_{1}^{n+1} \frac{1}{x} dx = \ln(n + 1)$

Adding 1 to both sides, we have that $H_n < 1 + \ln(n + 1)$, which means $y_n < 1$.

So $\{y_n\}$ is bounded. To show it is increasing, we'll prove $y_n - y_{n-1} > 0$.

$y_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \ln(n + 1)

y_{n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-2} - \ln(n)

y_n - y_{n-1} = \frac{1}{n} - [\ln(n + 1) - \ln(n)]

We can show that $y_n - y_{n-1} > 0$ by comparing areas:

$\frac{1}{n} = \text{area of rectangle} > \int_{n}^{n+1} \frac{1}{x} dx = \ln(n + 1) - \ln(n)$

Therefore, $\{y_n\}$ is a bounded, increasing sequence. So it must converge to some number. Step 1 is concluded.
Step 2:
It isn’t difficult to prove, using methods similar to what I’ve already shown, that the sequence \( (H_n - \ln(n)) \) also converges.

Since both \( (H_n - \ln(n)) \) and \( (H_n - \ln(n + 1)) \) converge, and the difference of their \( n \)th terms approaches 0 as \( n \) approaches \( \infty \), then they must have the same limit. Call this number \( \gamma \).

\[
\gamma = \lim_{n \to \infty} (H_n - \ln(n + 1)) = \lim_{n \to \infty} (H_n - \ln(n))
\]

Step 3: Prove that \( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \ln(2) \)

Proof: Let \( S_{2n} \) be the sum of the first \( 2n \) terms of the alternating harmonic.

\[
S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n} = \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n}\right) - 2 \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)
\]

Note: \( S_{2n} = H_{2n} - H_n \)

\[
S_{2n} = \left[\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n}\right) - \ln(2n)\right] - \left[\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{n}\right) - \ln(n)\right] + \ln(2)
\]

Thus \( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \lim_{n \to \infty} S_{2n} = \gamma - \gamma + \ln(2) = \ln(2) \).

To calculate \( H_n = \sum_{k=1}^{n} \frac{1}{k} \) with \( n = 100,000 \), I used three methods and my TI-84:

Using the sum command, the calculator took 14 minutes. (Ans: 12.09014613)

Using the integral \( \int_{1}^{n} \frac{1}{x^2} \, dx \), the calculator took 12 seconds. (Ans: 12.09014613)

Geometric, with first term \( a = 1 \) and common ratio \( r = x \):

\[
1 - x^n = 1 + x + x^2 + \cdots + x^{n-1}
\]

Integrate both sides:

\[
\int_{1}^{\infty} \frac{1 - x^n}{1 - x} \, dx = \int_{1}^{\infty} (1 + x + x^2 + \cdots + x^{n-1}) \, dx
\]

\[
[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}]_1^\infty = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}
\]

With the approximation \( H_n = \ln(n) + 0.57721566 \), it was instantaneous. (12.090144112)

The green area sums to \( y = \lim_{n \to \infty} (H_n - \ln(n + 1)) \). (Wolfram-alpha)

The Euler-Mascheroni constant \( \gamma \) is approximately 0.57721566.

It is suspected to be irrational, but this has not yet been proven.

We can use \( S_{2n} = H_{2n} - H_n \) from our previous proof to prove that the harmonic series diverges.

Assume that the harmonic series converges. Then:

\[
0 = \lim_{n \to \infty} H_{2n} - \lim_{n \to \infty} H_n
\]

\[
\lim_{n \to \infty} H_{2n} = \lim_{n \to \infty} S_{2n} = \ln(2)
\]

By contradiction, the harmonic series diverges.

Telescoping Series

Example: \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = 1 \)

Classic Proof:

\[
S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}
\]

\[
= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)
\]

\[
= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)
\]

\[
= 1 - \frac{1}{n+1}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \to \infty} S_n = 1
\]
Ah, but here comes the problem! I put that last example on a test, and the inevitable happens:

\[
\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots
\]

\[
= \frac{1}{2}
\]

Let’s do that last summation correctly, using partial sums:

\[
S_n = \sum_{k=1}^{n} \left(\frac{k+2}{2k} - \frac{k+3}{2k+2}\right)
\]

\[
= \frac{2}{(n+1)(n+2)}
\]

\[
\lim_{n \to \infty} S_n = \frac{3}{2} - \frac{1}{2} = 1
\]

Problem: Take any geometric series, \(r \neq 1\), and turn it into a telescoping series.

Solution:

\[
S_n = a + ar + ar^2 + \ldots + ar^{n-1}
\]

\[
= a \left(\frac{1-r^n}{1-r}\right)
\]

In general, any geometric series, \(r \neq 1\), can be written as a telescoping series:

\[
\sum_{n=1}^{\infty} ar^{n-1} = \frac{a(1-r)}{1-r} = \frac{a}{1-r} \sum_{n=1}^{\infty} (r^n - r^{n+1})
\]
Selected References

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Thanks for attending!

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