Teaching Calculus in the 21st Century

Robert Cappetta, PhD
University of Illinois at Chicago
Professor Emeritus: College of DuPage
The fact is that the state of most students’ conceptual knowledge of mathematics after they have taken a calculus course is abysmal.

Non-Trivial Problems

For which values of $b$ does the following converge?

$$\sum_{n=1}^{\infty} b^{\ln n}$$
Non-Trivial Problem

Find a value of $a$ so that the following exists and then evaluate the limit.

$$\lim_{{x \to 3}} \frac{2x^2 - 2ax + x - a - 1}{x^2 - 2x - 3}$$

Annie Selden
Important Topics or Obsolete?
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• Curve sketching with mins, maxs, asymptotes, intercepts, points of inflection etc.
• Graphing Polar Equations by Hand
Discussion Question

What are the major challenges to teaching and learning calculus and how have these challenges changed in recent years?
Myth of Multitasking

• I am a great multitasker

• 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

https://www.psychologytoday.com/blog/creativity-without-borders/201405/the-myth-multitasking
Cell Phones

The use of cell phones in the classroom causes distraction to the class (Burns & Lohenry, 2010; Campbell, 2006; Gilroy, 2004; Shelton et al., 2009; Tindell & Bohlander, 2012).

Cell Phones

• More than 90% of students sent text messages during classes. Tindell and Bohlander (2012)
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• Cell phones distract 89% of students during study hours and nearly 77% of students during class. Potharst (2010)
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• Student performance improved once cell phones were banned. Bugeja (2007)
Challenges
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• Poor Student Preparation and Placement
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• AP Calculus
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• Divergent Teaching Philosophies
  – Calculus Reform vs Traditional Calculus
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• Students Need Motivation
Poor Student Preparation

• National Assessment of Educational Progress - NAEP
• Program for International Student Assessment - PISA
• ACT
National Assessment of Educational Progress

Proficient or Higher
25% of the overall population
  32% of Caucasian students
  7% of African American students
  12% of Hispanic students

No significant change in the overall number over the last ten years

https://www.nationsreportcard.gov/
Program for International Student Assessment

The PISA is a worldwide exam administered every three years that measures 15-year-olds in 72 countries. About 540,000 students took the exam in 2015.

US saw an 11 point drop in mathematics

ACT

• 41% of students scored a 22 or above on the mathematics section, indicating a 75% likelihood of passing college algebra.

University of Illinois ALEKS Study

• ACT poor placement tool
• ALEKS is adaptive and free-response
• It defines “Knowledge Spaces”
  – Real Numbers
  – Exponential and Logarithms
  – Functions; Trigonometry
  – Rational Expressions; Radical Expressions; Equations and Inequalities, Exponents and Polynomials.

http://www.tandfonline.com/doi/abs/10.1080/10511970.2013.801378
Preparation for Calculus

Figure 13. ALEKS Assessment Subscore differences from completion of PreCalc, 2008 to 2009

(A) Before PreCalc

(B) After PreCalc
Can This Model Apply Elsewhere?
Can This Model Apply Elsewhere?

- Students are not monitored as they take the assessments.
Can This Model Apply Elsewhere?

• Students are not monitored as they take the assessments.

• Students complete the process several weeks, if not months, before the start of the term.
Can This Model Apply Elsewhere?

• Costs are much higher for this placement model than previously used ones.
Can This Model Apply Elsewhere?

• Costs are much higher for this placement model than previously used ones.

• Is it politically feasible to make students take a placement test after completing the prerequisite course?
Discussion Question

• How does AP Calculus affect calculus teaching and learning at the collegiate level?
Joseph G. Rosenstein, J and Ahluwalia, A: Putting Brakes on the Rush to AP Calculus
AP Calculus

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• About half of the students who take calculus in high school take AP Calculus
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• About half of those who take the AP Calculus exam receive grades of 4 or 5

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• About half of those who take the AP Calculus course take the AP Calculus exam
• About half of those who take the AP Calculus exam receive grades of 4 or 5
• About half of those who receive grades of 4 or 5 actually continue on an accelerated path.

Joseph G. Rosenstein, J and Ahluwalia, A: Putting Brakes on the Rush to AP Calculus
Conclusions

• A very small percentage of those who are accelerated throughout high school maintain that acceleration through their first year at college.

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• There is no evidence that encouraging more students to take AP Calculus will expand the STEM pipeline.

Joseph G. Rosenstein, J and Ahluwalia, A: Putting Brakes on the Rush to AP Calculus
Recommendations

• Students should be encouraged to take an AP Calculus course only if they
  – received an A or B+ in the Precalculus course
  – indicate that they like mathematics and wish to be challenged
  – intend to take more advanced math courses in their first year in college
  – Intend to pursue studies in fields the require numerous collegiate math classes.

Joseph G. Rosenstein, J and Ahluwalia, A: Putting Brakes on the Rush to AP Calculus
Mainstream Calculus I Enrollments
(fall only for 2- & 4-yr colleges & universities)

students (thousands)

academic year

<table>
<thead>
<tr>
<th>Exam Score</th>
<th>Calculus AB</th>
<th>Calculus BC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td>% At</td>
</tr>
<tr>
<td>5</td>
<td>76,486</td>
<td>24.8</td>
</tr>
<tr>
<td>4</td>
<td>53,467</td>
<td>17.3</td>
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<tr>
<td>3</td>
<td>53,533</td>
<td>17.4</td>
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<tr>
<td>2</td>
<td>30,017</td>
<td>9.7</td>
</tr>
<tr>
<td>1</td>
<td>94,712</td>
<td>30.7</td>
</tr>
<tr>
<td>Number of Students</td>
<td>308,215</td>
<td></td>
</tr>
<tr>
<td>3 or Higher / %</td>
<td>183,486</td>
<td>59.5</td>
</tr>
<tr>
<td>Mean Score</td>
<td>2.96</td>
<td></td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>1.58</td>
<td></td>
</tr>
</tbody>
</table>
Discussion Question

• What are some of the differing viewpoints held by college calculus instructors?
Four Philosophies

• A process-oriented view in which mathematics is defined as a heuristic and creative activity that allows solving problems using different and individual ways.

• An application-oriented view that accentuates the utility of mathematics for the real world. Application-oriented goals are part of the *modelling trend*. 

Four Philosophies

• A formalist (world) view in which mathematics is characterized by a strongly logical and formal approach and in which accuracy and precision are important.

• A schema view in which mathematics is seen as a collection of calculation rules and procedures to be memorized and applied in routine tasks.

Goals

• Most important goal is that students automate the processes of differentiation and integration.

• Most important goal is that students understand differentiation and integration in a formal way.
Goals

• Most important goal is that students realize the relationship between the topics of calculus and their daily lives.

• Most important goal is that students are able to re-invent the main concepts of calculus in their individual ways.

Calculus Reform

Dubinsky - Main issue is not the topics in calculus but how it is taught

– the role of a teacher is not to explain mathematics in a classroom, but to induce students to construct it in their minds.

– Use computers to write code so that students see mathematical concepts as processes, objects and schemas.

– Collaborative learning used to promote reflection.

http://www.math.kent.edu/~edd/CasePaper.pdf
Calculus Reform

• Every topic should be presented using three approaches: graphical (use of pictures, diagrams, and/or graphs), numerical (tables of values) and algebraic (equations and formulas).

• Formal definitions and procedures evolve from the investigation of practical problems (the Way of Archimedes)

AN EVALUATION OF REFORM IN THE TEACHING OF CALCULUS
Juan Cadena, Betty Travis, and Sandy Norman
Conclusions

• All instructors reported using guided discovery learning techniques and cooperative learning in the context of small groups.
• Significantly more students earned an A or a B than expected in the reform section.
• Students who initially take a traditional Calculus I course fare no better, in subsequent calculus based mathematics courses than do those students who have taken a reform Calculus.
Critics

As Friedman (1993, p. 7) states in his investigation of calculus reform, "It is ironic that mathematicians who are trained never to accept a mathematical proposition without proof are willing to accept a proposition in another discipline without proof. We know what constitutes a proof in mathematics. What constitutes a proof in the teaching of mathematics?"
Discussion Question

• What are the best ways to use technology in college-level calculus?
Primary Role of Technology

• Facilitating computation
• Checking solutions
• Making conjectures
• Multiple representations

Would Wolfram Alpha Pass Your Calculus Exam?
Input interpretation:
\[
\sum_{n=1}^{\infty} \frac{(-3)^n}{n + 1}
\]

Result:
\[
\sum_{n=1}^{\infty} \frac{(-3)^n}{n + 1} \quad \text{(sum does not converge)}
\]

Convergence tests:
By the limit test, the series diverges.
Possible intermediate steps:

\[ \int \sin^{-1}(3x) \, dx \]

For the integrand \( \sin^{-1}(3x) \), substitute \( u = 3x \) and \( du = 3 \, dx \):

\[ = \frac{1}{3} \int \sin^{-1}(u) \, du \]

For the integrand \( \sin^{-1}(u) \), integrate by parts, \( \int f \, dg = fg - \int g \, df \), where

\[ f = \sin^{-1}(u), \quad dg = du, \]
\[ df = \frac{1}{\sqrt{1-u^2}} \, du, \quad g = u: \]

\[ = \frac{1}{3} u \sin^{-1}(u) - \frac{1}{3} \int \frac{u}{\sqrt{1-u^2}} \, du \]

For the integrand \( \frac{u}{\sqrt{1-u^2}} \), substitute \( s = 1 - u^2 \) and \( ds = -2udu \):

\[ = \frac{1}{6} \int \frac{1}{\sqrt{s}} \, ds + \frac{1}{3} u \sin^{-1}(u) \]
The integral of $\frac{1}{\sqrt{s}}$ is $2 \sqrt{s}$:

$$= \frac{\sqrt{s}}{3} + \frac{1}{3} u \sin^{-1}(u) + \text{constant}$$

Substitute back for $s = 1 - u^2$:

$$= \frac{\sqrt{1-u^2}}{3} + \frac{1}{3} u \sin^{-1}(u) + \text{constant}$$

Substitute back for $u = 3x$:

$$= \frac{1}{3} \sqrt{1 - 9x^2} + x \sin^{-1}(3x) + \text{constant}$$
Check the derivative of a solution

\[ \int 5^x \, dx = \frac{5^x}{\ln 5} + C \]

\begin{verbatim}
Plot1 Plot2 Plot3
\texttt{\textbackslash Y_1}=(5^\texttt{\textbackslash X})/(/(\ln(5))
)+3
\texttt{\textbackslash Y_2} \texttt{\textbullet n}Deriv(\texttt{Y_1}, \texttt{X},
\texttt{X}, .001)
\texttt{\textbackslash Y_3} \texttt{\textbullet 5}^\texttt{\textbackslash X}
\texttt{\textbackslash Y_4}=
\texttt{\textbackslash Y_5}=
\end{verbatim}
Harmonic Series

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]
Program

PROGRAM: SUMTOTAL
: 0→Z
: Prompt T
: For(N, 1, T, 1)
: 1/N+Z→Z
: End
: Z
Program SUMTOTAL

T=10:
2.928968254
T=100:
5.187377518
T=1000:
7.485470861

7.485470861
T=10000:
9.787606036
T=20000:
10.48072822
T=30000:
10.88618499
sum \( \frac{1}{n} \), \( n=1..10^{18} \)

Approximated sum:

\[
\sum_{n=1}^{1000000000000000} \frac{1}{n} \approx 42.0237
\]

Partial sums:
\[ f(x, y) = x^4 + y^4 = 2xy \]
Discussion Question

Describe the types of questions are usually included on final exams in college-level calculus classes.
Final Exam Study

Final exams generally require low levels of cognitive demand, seldom contain problems stated in a real-world context, rarely elicit explanation, and do not require students to demonstrate or apply their understanding of the course’s central ideas.

Six Intellectual Behaviors

• Remember
• Recall and Apply Procedure
• Understand
• Apply Understanding
• Analyze
• Evaluate
• Create

## Calculus Final Exam Analysis

<table>
<thead>
<tr>
<th>Item Orientation</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Remember</td>
<td>6.51</td>
</tr>
<tr>
<td>Recall and Apply Procedure</td>
<td>78.70</td>
</tr>
<tr>
<td>Understand</td>
<td>4.42</td>
</tr>
<tr>
<td>Apply Understanding</td>
<td>10.30</td>
</tr>
<tr>
<td>Analyze</td>
<td>0.11</td>
</tr>
<tr>
<td>Evaluate</td>
<td>0</td>
</tr>
<tr>
<td>Create</td>
<td>0</td>
</tr>
</tbody>
</table>

Recommendations

• Identify the barriers posed by student misunderstandings of fundamental concepts.
• Recognize the importance of non-cognitive factors such as motivation and self-efficacy.
• Modify assignments in Calculus I in ways that promote higher-order thinking.
• Collaborate in the process of designing assessments of outcomes.

Discussion Question

• What are the evolving political and cultural forces that may influence the way calculus is taught and learned?
Changing Political Realities

• Fewer tenured faculty members
• More management control of curriculum
• Pressure to improve retention/student success numbers
• Increased importance of student evaluations
• Rate My Professor/Social Media

CBMS (2010)
Political Realities

• Decreased funding for public higher education.
• Students are less well-prepared for college-level mathematics.
• Politicians stress “accountability” which often means improving student performance while decreasing resources.
Quality Imperative

• the quality shortfall is just as urgent as the attainment shortfall (American Association of Colleges and Universities – 2010)
• too many students are making little or no progress on important learning outcomes while in college
• the increasing complexity of our world is adding to what a well-educated person must know and be able to do
Employer Demands (Schneider 2010)

• Communications
• Analytical Reasoning
• Quantitative Literacy
• Broad knowledge of science and society
• Field-specific knowledge and skills
• Global knowledge and competence
• Teamwork and problem solving skills in diverse settings
• Information Literacy and Fluency
• Ethical Reasoning and Decision Making
Motivate Students

• Be comfortable using humor in the classroom. (Weaver and Cotrell, 1987)
• Have high expectations for students – (Jaime Escalante)
• Teachers must be comfortable in the tasks completed during the lecture.
• Teaching should be interactive.
• Competition and collaboration among students may increase motivation

http://math.coe.uga.edu/TME/Issues/v03n2/Eggleton.pdf
Motivation

• The concern shown by teachers in taking the time to find the techniques that best fit the class and allow the teachers to feel comfortable and professional is essential to motivating students.

http://math.coe.uga.edu/TME/Issues/v03n2/Eggleton.pdf
How to Motivate Adult Learners

• Focus on short-term goals
• Emphasize that all progress matters
• Let them know it’s okay to get it wrong
• Make it relevant to their lives
• Tap into their intrinsic motivation

https://www.seedsofliteracy.org/5-ways-to-motivate-adult-learners/
\[ z = b^{\ln n} \]

\[ \ln z = \ln b^{\ln n} \]

\[ \ln z = (\ln n)(\ln b) \]
\[ z = b^{\ln n} \]

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\[ \ln z = (\ln n)(\ln b) \]

\[ z = e^{(\ln n)(\ln b)} \]

\[ z = n^{\ln b} \]

\[ z = \frac{1}{n^{-\ln b}} \]
\[ z = b^{\ln n} \]
\[ \ln z = \ln b^{\ln n} \]
\[ \ln z = (\ln n)(\ln b) \]
\[ z = e^{(\ln n)(\ln b)} \]
\[ z = n^{\ln b} \]
\[ z = \frac{1}{n^{-\ln b}} \]
\[ -\ln b > 1 \]
\[ \ln b < -1 \]
\[ b < e^{-1} \]
\[ b < \frac{1}{e} \]
\[
\lim_{x \to 3} \frac{2x^2 - 2ax + x - a - 1}{x^2 - 2x - 3}
\]

If \(20-7a \neq 0\), the limit will not exist.

Let \(a = \frac{20}{7}\) and then show that the limit exists.

Plugging in 3 for \(x\), we get

\[
\frac{2(3)^2 - 2a(3) + 3 - a - 1}{(3)^2 - 2(3) - 3} = \frac{18 - 6a + 3 - a - 1}{9 - 6 - 3}
\]

\[
\frac{20 - 7a}{0}
\]
\[
\lim_{{x \to 3}} \frac{2x^2 - 2ax + x - a - 1}{x^2 - 2x - 3} = \\
\lim_{{x \to 3}} \frac{2x^2 - 2 \left( \frac{20}{7} \right) x + x - \frac{20}{7} - 1}{x^2 - 2x - 3} = \\
\lim_{{x \to 3}} \frac{2x^2 - \left( \frac{40}{7} \right) x + x - \frac{27}{7}}{x^2 - 2x - 3} = \\
\lim_{{x \to 3}} \frac{2x^2 - \left( \frac{33}{7} \right) x - \frac{27}{7}}{x^2 - 2x - 3} = \\
\lim_{{x \to 3}} \frac{14x^2 - 33x - 27}{7(x^2 - 2x - 3)} = \\
\lim_{{x \to 3}} \frac{(14x + 9)(x - 3)}{7(x - 3)(x + 1)} = \\
\lim_{{x \to 3}} \frac{14x + 9}{7(x + 1)} = \\
\frac{14(3) + 9}{7(3+1)} = \frac{42 + 9}{7 \cdot 4} = \frac{51}{28}
\]
Thank you

• Robert Cappetta
• <a href="mailto:cappetta@uic.edu">cappetta@uic.edu</a>
• Slides available at
  sites.google.com/site/RWCAMATYC/files
• Look for AMATYC2017.pdf