# Teaching Calculus in the 21st Century

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# Susanna Epp (1987)

The fact is that the state of most students' conceptual knowledge of mathematics after they have taken a calculus course is abysmal.

Epp, S. (1987). The logic of teaching calculus. In R. G. Douglas (Ed.), *Toward a lean and lively calculus* (pp. 41-60). Washington, D.C.: Mathematical Association of America.

# Non-Trivial Problems

For which values of b does the following converge?

$$\sum_{n=1}^{\infty} b^{\ln n}$$

# Non-Trivial Problem

Find a value of *a* so that the following exists and then evaluate the limit.

$$\lim_{x \to 3} \frac{2x^2 - 2ax + x - a - 1}{x^2 - 2x - 3}$$

Annie Selden

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- Curve sketching with mins, maxs, asymptotes, intercepts, points of inflection etc.
- Graphing Polar Equations by Hand

# Discussion Question

What are the major challenges to teaching and learning calculus and how have they these challenges changed in recent years?

# Myth of Multitasking

- I am a great multitasker
- 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

https://www.psychologytoday.com/blog/
creativity-without-borders/201405/the-myth-multitasking

# Cell Phones

The use of cell phones in the classroom causes distraction to the class (Burns & Lohenry, 2010; Campbell, 2006; Gilroy, 2004; Shelton et al., 2009; Tindell & Bohlander, 2012).

Ali, A. I., Papakie, M. R., & McDevitt, T. (2012). Dealing with the distractions of cell phone Misuse/Use in the classroom - A case example. *Competition Forum, 10*(2), 220-230.

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- Student performance improved once cell phones were banned. Bugeja (2007)

# Challenges

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- Quality of Exams
- Changing Political Realities
- Students Need Motivation

# Poor Student Preparation

- National Assessment of Educational Progress - NAEP
- Program for International Student Assessment - PISA
- ACT

# National Assessment of Educational Progress

Proficient or Higher

25% of the overall population

32% of Caucasian students

7% of African American students

12% of Hispanic students

No significant change in the overall number over the last ten years

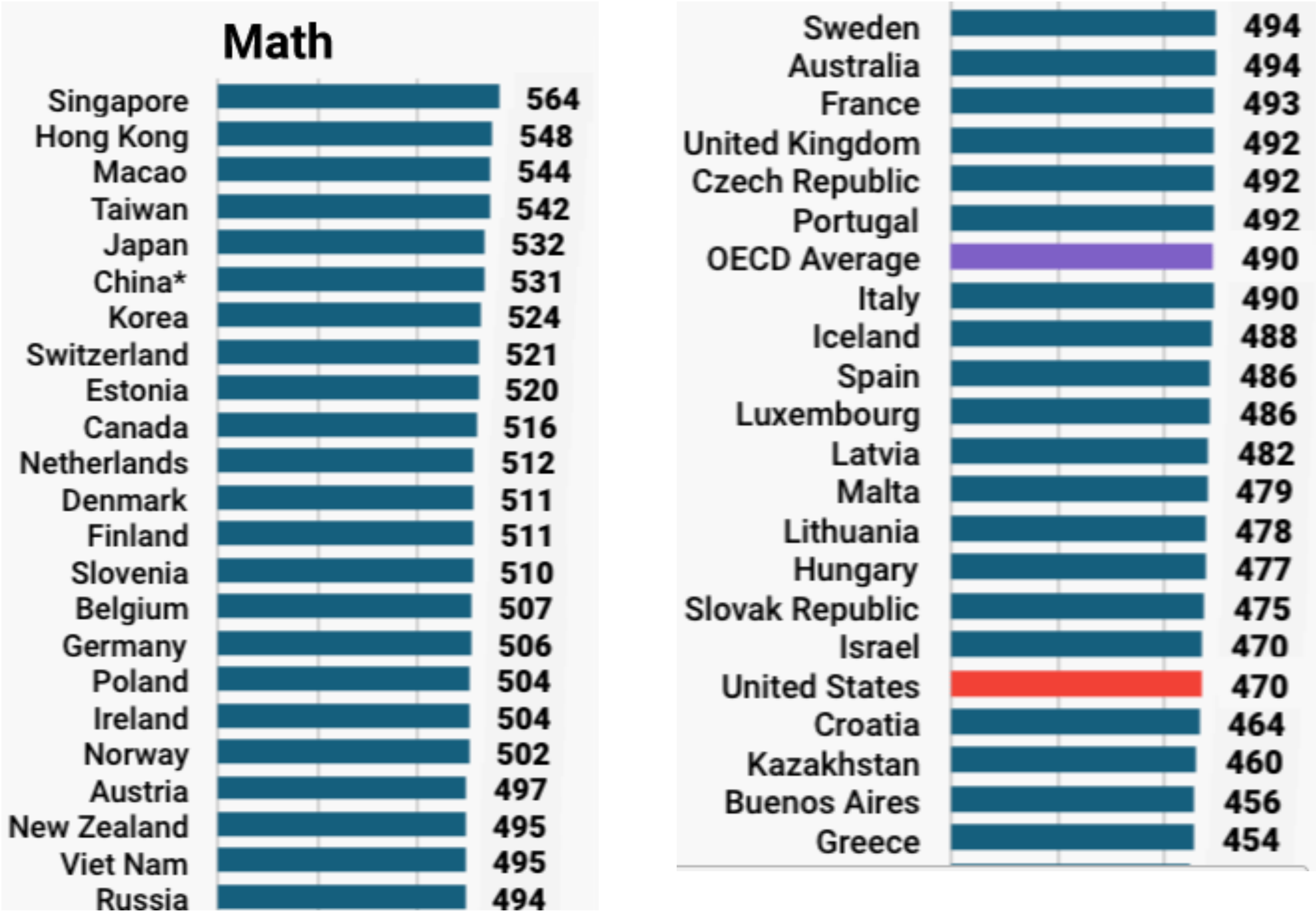
https://www.nationsreportcard.gov/

# Program for International Student Assessment

The PISA is a worldwide exam administered every three years that measures 15-year-olds in 72 countries. About 540,000 students took the exam in 2015.

US saw an 11 point drop in mathematics

http://www.businessinsider.com/pisa-worldwide-ranking-of-math-science-reading-skills-2016-12




http://www.businessinsider.com/pisa-worldwide-ranking-of-math-science-reading-skills-2016-12

# ACT

- 41% of students scored a 22 or above on the mathematics section, indicating a 75% likelihood of passing college algebra.

https://www.usnews.com/news/politics/articles/2016-08-24/bigger-numbers-of-high-school-grads-taking-act-college-test

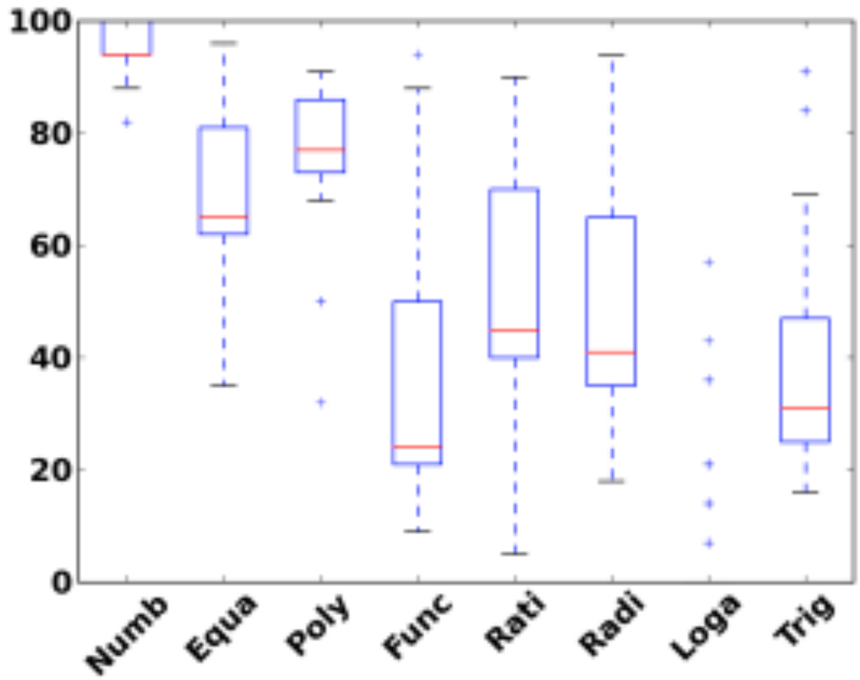
# University of Illinois ALEKS Study

- ACT poor placement tool
- ALEKS is adaptive and free-response
- It defines "Knowledge Spaces"
  - Real Numbers
  - Exponential and Logarithms
  - Functions; Trigonometry
  - Rational Expressions; Radical Expressions; Equations and Inequalities, Exponents and Polynomials.

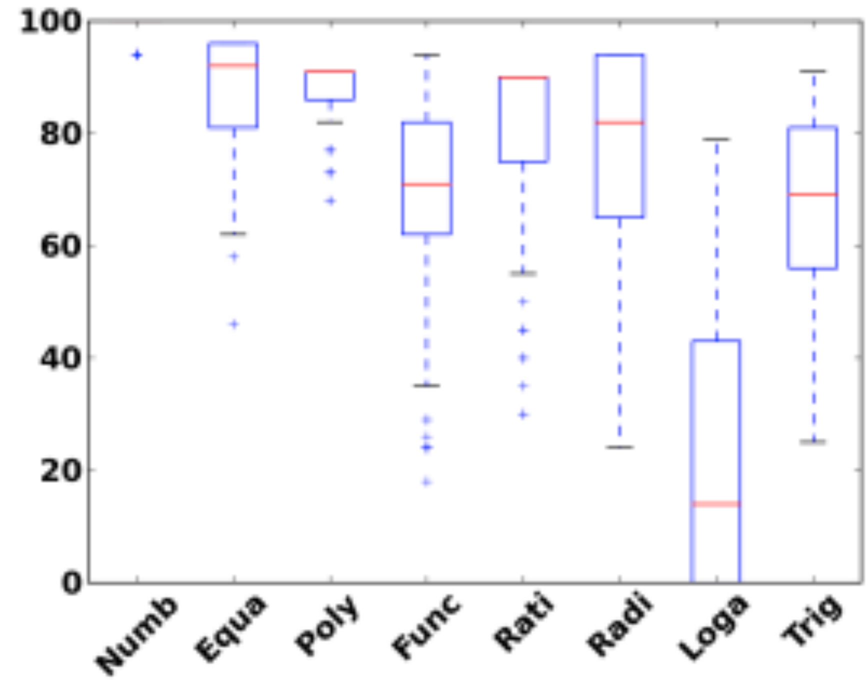
http://www.tandfonline.com/doi/abs/10.1080/10511970.2013.801378

# Preparation for Calculus

FIGURE 13. ALEKS Assessment Subscore differences from completion of PreCalc, 2008 to 2009




(A) Before PreCalc




(B) After PreCalc

# Can This Model Apply Elsewhere?

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- Students are not monitored as they take the assessments.

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- Students complete the process several weeks, if not months, before the start of the term.

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- Costs are much higher for this placement model than previously used ones.

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- Costs are much higher for this placement model than previously used ones.
- Is it politically feasible to make students take a placement test after completing the prerequisite course?

# Discussion Question

- How does AP Calculus affect calculus teaching and learning at the collegiate level?

# AP Calculus

Joseph G. Rosenstein, J and Ahluwalia, A: Putting Brakes on the Rush to AP Calculus

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- About half of those who receive grades of 4 or 5 actually continue on an accelerated path.

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# Conclusions

- A very small percentage of those who are accelerated throughout high school maintain that acceleration through their first year at college.

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- A very small percentage of those who are accelerated throughout high school maintain that acceleration through their first year at college
- There is no evidence that encouraging more students to take AP Calculus will expand the STEM pipeline.

Joseph G. Rosenstein, J and Ahluwalia, A: Putting Brakes on the Rush to AP Calculus

# Recommendations

- Students should be encouraged to take an AP Calculus course only if they
  - received an A or B+ in the Precalculus course
  - indicate that they like mathematics and wish to be challenged
  - intend to take more advanced math courses in their first year in college
  - Intend to pursue studies in fields the require numerous collegiate math classes.

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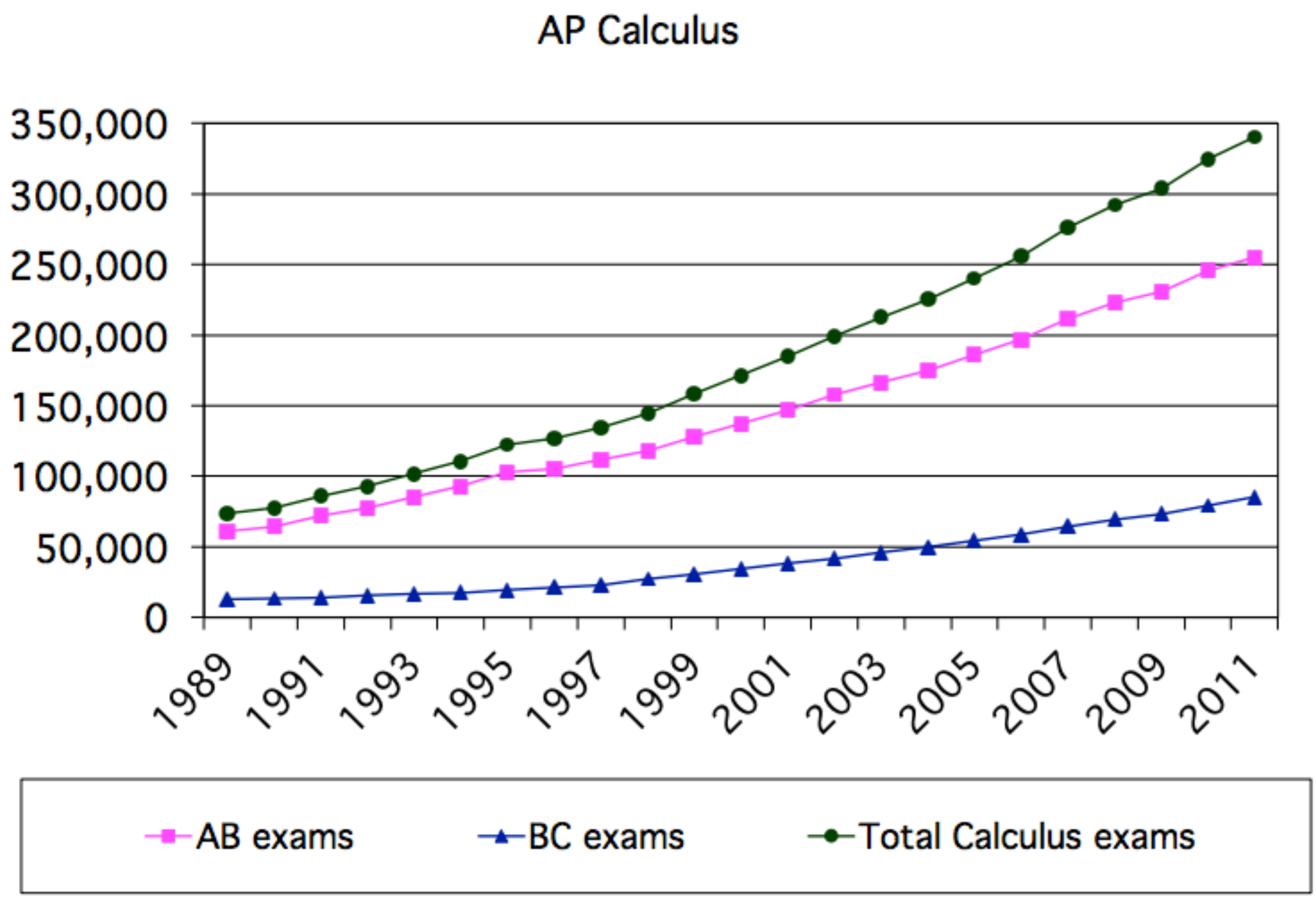
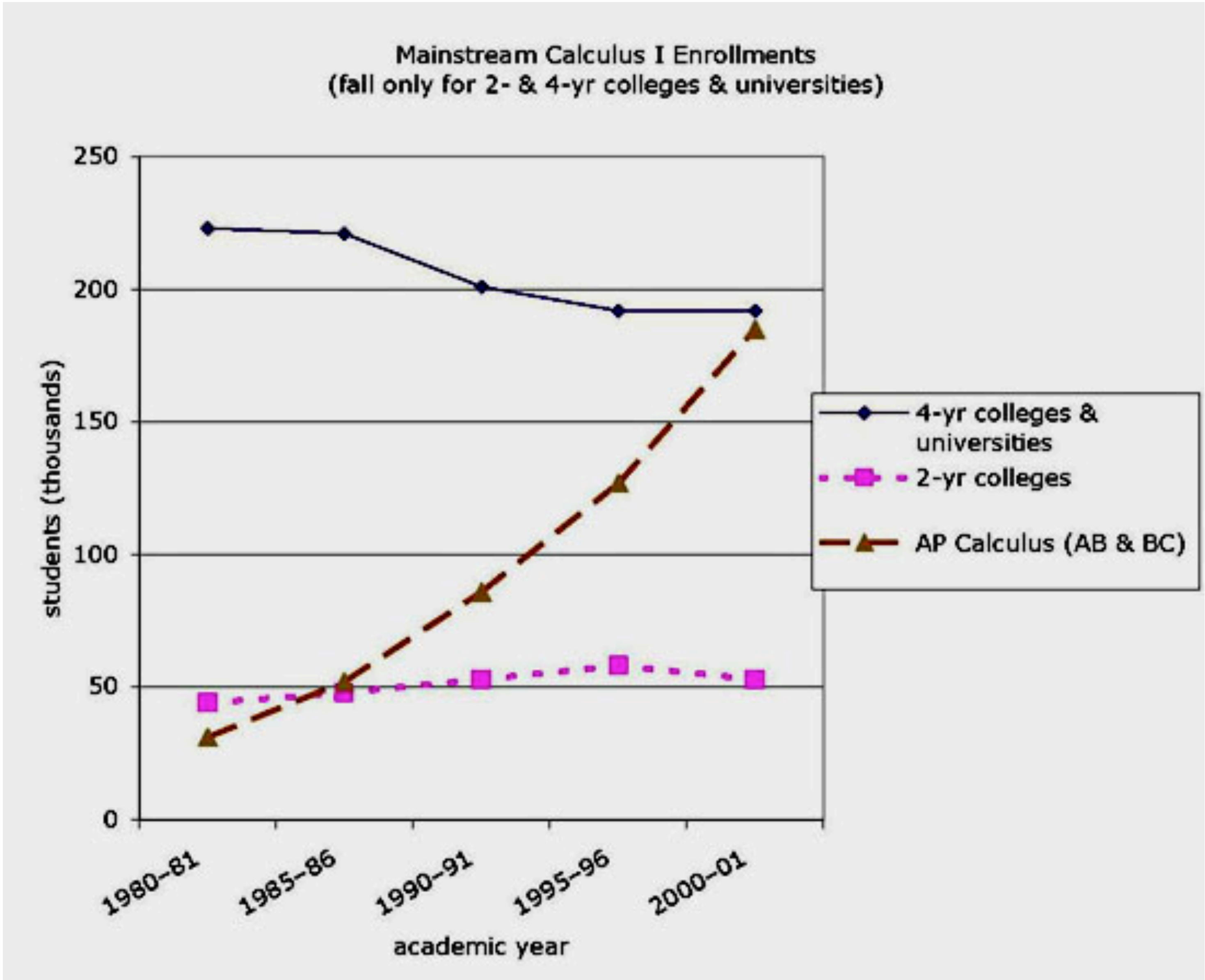



https://www.macalester.edu/~bressoud/talks/2012/Iowa-APCalc.pdf




http://www.maa.org/the-changing-face-of-calculus-first-semester-calculus
-as-a-high-school-course

# STUDENT SCORE DISTRIBUTIONS*

## AP Exams - May 2016

| Exam Score | Calculus AB | | Calculus BC | |
|---|---|---|---|---|
| | N | % At | N | % At |
| 5 | 76,486 | 24.8 | 60,632 | 48.5 |
| 4 | 53,467 | 17.3 | 19,191 | 15.4 |
| 3 | 53,533 | 17.4 | 21,441 | 17.2 |
| 2 | 30,017 | 9.7 | 7,212 | 5.8 |
| 1 | 94,712 | 30.7 | 16,455 | 13.2 |
| Number of Students | 308,215 | | 124,931 | |
| 3 or Higher / % | 183,486 | 59.5 | 101,264 | 81.1 |
| Mean Score | 2.96 | | 3.80 | |
| Standard Deviation | 1.58 | | 1.43 | |

# Discussion Question

- What are some of the differing viewpoints held by college calculus instructors?

# Four Philosophies

- A process-oriented view in which mathematics is defined as a heuristic and creative activity that allows solving problems using different and individual ways.
- An application-oriented view that accentuates the utility of mathematics for the real world. Application-oriented goals are part of the *modelling trend*.

# Four Philosophies

- A formalist (world) view in which mathematics is characterized by a strongly logical and formal approach and in which accuracy and precision are important.
- A schema view in which mathematics is seen as a collection of calculation rules and procedures to be memorized and applied in routine tasks.

Grigutsch, S., Raatz, U., & Törner, G. (1998). Einstellungen gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematikdidaktik, 19*(1), 3–45.

# Goals

- Most important goal is that students automate the processes of differentiation and integration.
- Most important goal is that students understand differentiation and integration in a formal way.

# Goals

- Most important goal is  that students realize the relationship between the topics of calculus and their daily lives.
- Most important goal is that students are able to re-invent the main concepts of calculus in their individual ways.

Eichler, A. & Erens, R. ZDM Mathematics Education (2014) 46: 647
. doi:10.1007/s11858-014-0606-y

# Calculus Reform

Dubinsky - Main issue is not the topics in calculus but how it is taught

- the role of a teacher is not to explain mathematics in a classroom, but to induce students to construct it in their minds.
- Use computers to write code so that students see mathematical concepts as processes, objects and schemas.
- Collaborative learning used to promote reflection.

http://www.math.kent.edu/~edd/CasePaper.pdf

# Calculus Reform

- Every topic should be presented using three approaches: graphical (use of pictures, diagrams, and/or graphs), numerical (tables of values) and algebraic (equations and formulas).
- Formal definitions and procedures evolve from the investigation of practical problems (the Way of Archimedes)

AN EVALUATION OF REFORM IN THE TEACHING OF CALCULUS
Juan Cadena, Betty Travis, and Sandy Norman

# Conclusions

- All instructors reported using guided discovery learning techniques and cooperative learning in the context of small groups.
- Significantly more students earned an A or a B than expected in the reform section.
- Students who initially take a traditional Calculus I course fare no better, in subsequent calculus based mathematics courses than do those students who have taken a reform Calculus.

AN EVALUATION OF REFORM IN THE TEACHING OF CALCULUS
Juan Cadena, Betty Travis, and Sandy Norman

# Critics

As Friedman (1993, p. 7) states in his investigation of calculus reform, "It is ironic that mathematicians who are trained never to accept a mathematical proposition without proof are willing to accept a proposition in another discipline without proof. We know what constitutes a proof in mathematics. What constitutes a proof in the teaching of mathematics?"

# Discussion Question

- What are the best ways to use technology in college-level calculus?

# Primary Role of Technology

- Facilitating computation
- Checking solutions
- Making conjectures
- Multiple representations

Eichler, A. & Erens, R. ZDM Mathematics Education (2014) 46: 647
. doi:10.1007/s11858-014-0606-y

# Would Wolfram Alpha Pass Your Calculus Exam?

WolframAlpha computational knowledge engine.

sum (-3)^n/(n+1), n=1..infinity

Web Apps  Examples  Random

Input interpretation:

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n+1}$$

Result:

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n+1} \quad \text{(sum does not converge)}$$

Convergence tests:

By the limit test, the series diverges.

Possible intermediate steps:

$$\int \sin^{-1}(3x)\, dx$$

For the integrand $\sin^{-1}(3x)$, substitute $u = 3x$ and $du = 3dx$:

$$= \frac{1}{3} \int \sin^{-1}(u)\, du$$

For the integrand $\sin^{-1}(u)$, integrate by parts, $\int f\, dg = f g - \int g\, df$, where

$f = \sin^{-1}(u), \quad dg = du,$

$df = \frac{1}{\sqrt{1-u^2}}\, du, \quad g = u$:

$$= \frac{1}{3} u \sin^{-1}(u) - \frac{1}{3} \int \frac{u}{\sqrt{1-u^2}}\, du$$

For the integrand $\frac{u}{\sqrt{1-u^2}}$, substitute $s = 1-u^2$ and $ds = -2\, u du$:

$$= \frac{1}{6} \int \frac{1}{\sqrt{s}}\, ds + \frac{1}{3} u \sin^{-1}(u)$$

The integral of $\frac{1}{\sqrt{s}}$ is $2\sqrt{s}$ :

$$= \frac{\sqrt{s}}{3} + \frac{1}{3}\, u \sin^{-1}(u) + \text{constant}$$

Substitute back for $s = 1 - u^2$:

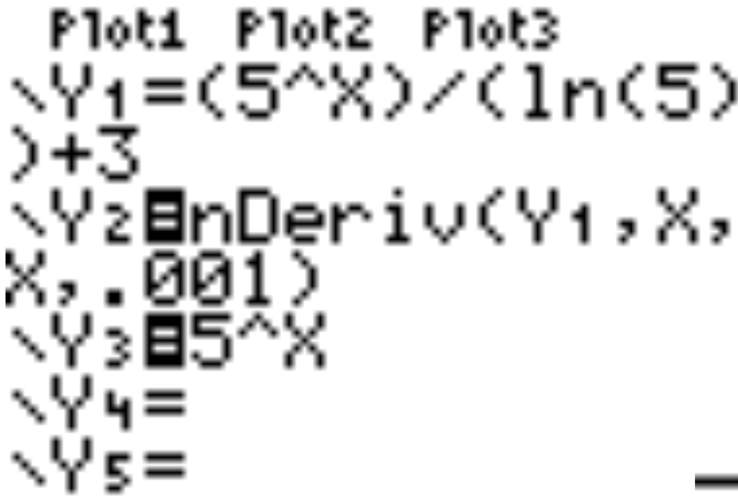
$$= \frac{\sqrt{1-u^2}}{3} + \frac{1}{3}\, u \sin^{-1}(u) + \text{constant}$$

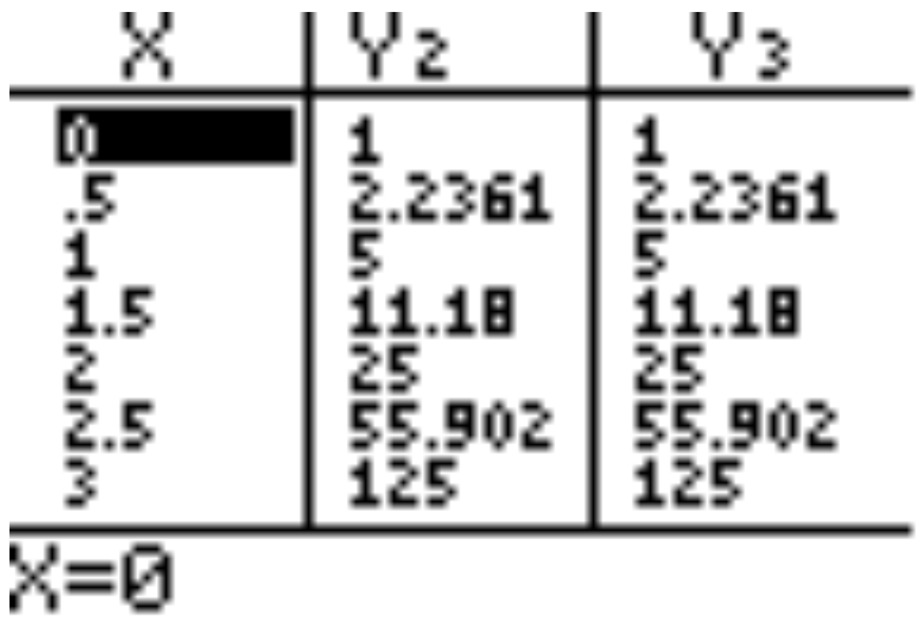
Substitute back for $u = 3\,x$:

$$= \frac{1}{3}\sqrt{1 - 9\,x^2} + x \sin^{-1}(3\,x) + \text{constant}$$

# Check the derivative of a solution

$$\int 5^x dx = \frac{5^x}{\ln 5} + C$$



| X | Y2 | Y3 |
|---|---|---|
| 0 | 1 | 1 |
| .5 | 2.2361 | 2.2361 |
| 1 | 5 | 5 |
| 1.5 | 11.18 | 11.18 |
| 2 | 25 | 25 |
| 2.5 | 55.902 | 55.902 |
| 3 | 125 | 125 |

X=0

# Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

# Program

```
PROGRAM:SUMTOTAL
:0→Z
:Prompt T
:For(N,1,T,1)
:1/N+Z→Z
:End
:Z
```

```
prgmSUMTOTAL
T=?10
      2.928968254
T=?100
      5.187377518
T=?1000
      7.485470861
█
```

```
      7.485470861
T=?10000
      9.787606036
T=?20000
      10.48072822
T=?30000
      10.88618499
█
```

WolframAlpha computational knowledge engine.

sum (1/n), n=1..10^18

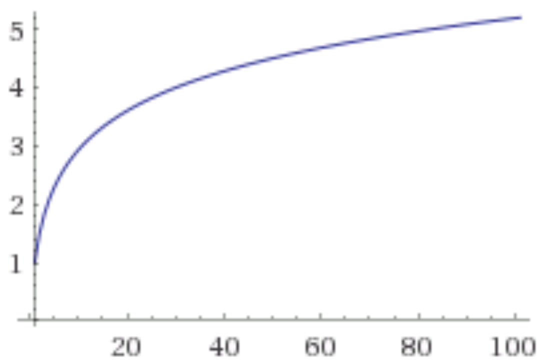
Web Apps Examples Random

Approximated sum: More digits

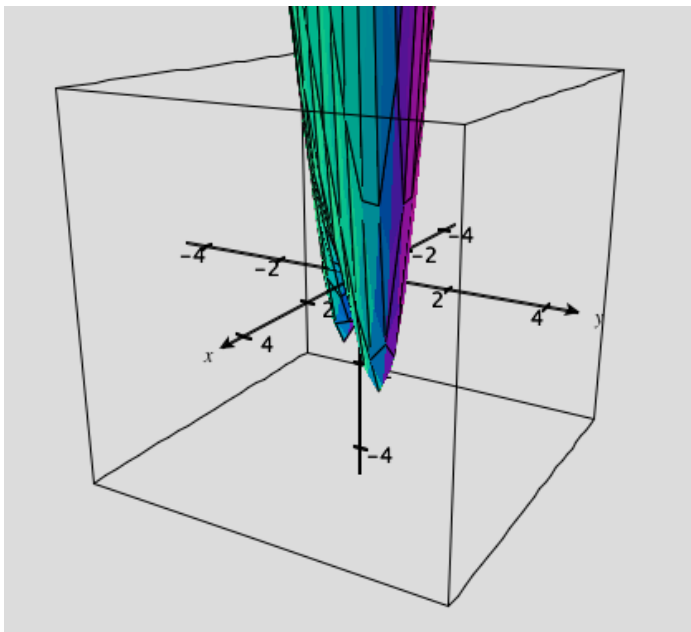
$$\sum_{n=1}^{1\,000\,000\,000\,000\,000\,000} \frac{1}{n} \approx 42.0237$$

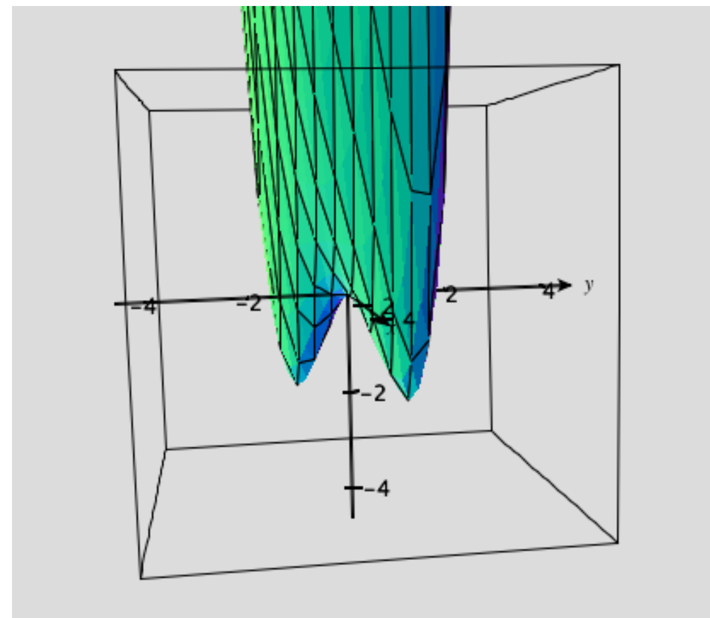
Open code

Partial sums:



$$f(x,y) = x^4 + y^4 = 2xy$$



# Discussion Question

Describe the types of questions are usually included on final exams in college-level calculus classes.

# Final Exam Study

Final exams generally require low levels of cognitive demand, seldom contain problems stated in a real-world context, rarely elicit explanation, and do not require students to demonstrate or apply their understanding of the course's central ideas.

Tallman, M., & Carlson, M. P. (2012). A characterization of calculus I final exams in U.S. colleges and universities. *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (p. 217–226). Portland, OR: Portland State University.

# Six Intellectual Behaviors

- Remember
- Recall and Apply Procedure
- Understand
- Apply Understanding
- Analyze
- Evaluate
- Create

Cited in Tallman, M., & Carlson, M. P. (2012) adapted from  Anderson and Krathwohl (2001)

# Calculus Final Exam Analysis

| Item Orientation | % |
|---|---|
| Remember | 6.51 |
| Recall and Apply Procedure | 78.70 |
| Understand | 4.42 |
| Apply Understanding | 10.30 |
| Analyze | 0.11 |
| Evaluate | 0 |
| Create | 0 |

Tallman, M., & Carlson, M. P. (2012).

# Recommendations

- Identify the barriers posed by student misunderstandings of fundamental concepts.
- Recognize the importance of non-cognitive factors such as motivation and self-efficacy.
- Modify assignments in Calculus I in ways that promote higher-order thinking.
- Collaborate in the process of designing assessments of outcomes.

Burn, H., & Mesa, V. (2015). The Calculus I curriculum. In D. M. Bressoud, V. Mesa, & C. L. Rasmussen (Eds.), *Insights and recommendations from the MAA National Study of College Calculus*. (pp. 45-57). Washington, DC.

# Discussion Question

- What are the evolving political and cultural forces that may influence the way calculus is taught and learned?

# Changing Political Realities

- Fewer tenured faculty members
- More management control of curriculum
- Pressure to improve retention/student success numbers
- Increased importance of student evaluations
- Rate My Professor/Social Media

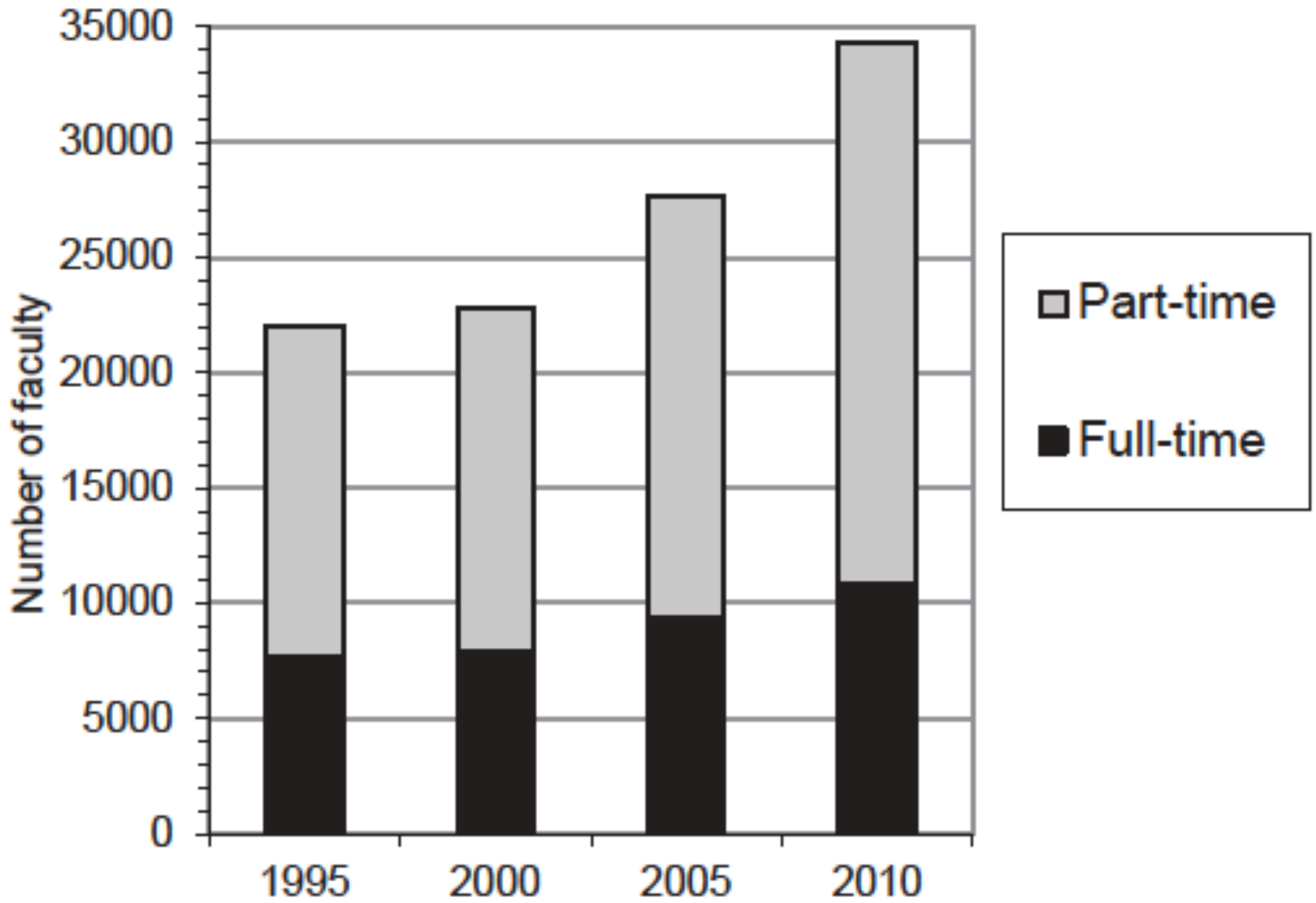



**FIGURE S.14.4** Number of full-time and part-time faculty in mathematics programs at two-year colleges in fall 1995, 2000, 2005, and 2010.

Conference Board of the Mathematical Sciences (2010)

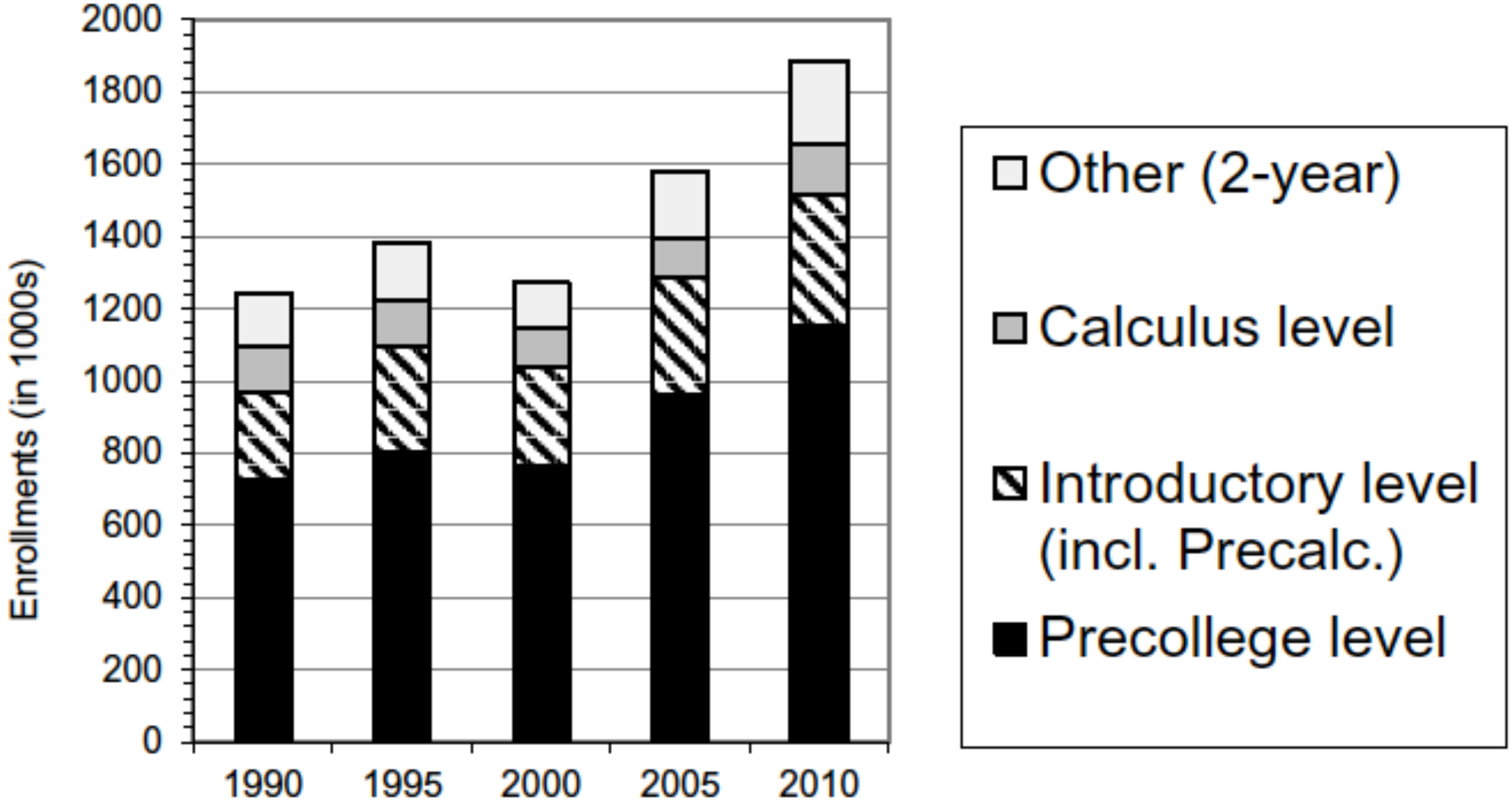



**FIGURE S.2.2** Enrollments (in 1000s) in undergraduate mathematics courses in two-year college mathematics programs by level of course in the fall of 1990, 1995, 2000, 2005, and 2010.

CBMS (2010)

# Political Realities

- Decreased funding for public higher education.
- Students are less well-prepared for college-level mathematics.
- Politicians stress "accountability" which often means improving student performance while decreasing resources.

# Quality Imperative

- the quality shortfall is just as urgent as the attainment shortfall (American Association of Colleges and Universities – 2010)
- too many students are making little or no progress on important learning outcomes while in college
- the increasing complexity of our world is adding to what a well-educated person must know and be able to do

# Employer Demands (Schneider 2010)

- Communications
- Analytical Reasoning
- Quantitative Literacy
- Broad knowledge of science and society
- Field-specific knowledge and skills
- Global knowledge and competence
- Teamwork and problem solving skills in diverse settings
- Information Literacy and Fluency
- Ethical Reasoning and Decision Making

# Motivate Students

- Be comfortable using humor in the classroom. (Weaver and Cotrell, 1987)
- Have high expectations for students – (Jaime Escalante)
- Teachers must be comfortable in the tasks completed during the lecture.
- Teaching should be interactive.
- Competition and collaboration among students may increase motivation

http://math.coe.uga.edu/TME/Issues/v03n2/Eggleton.pdf

# Motivation

- The concern shown by teachers in taking the time to find the techniques that best fit the class and allow the teachers to feel comfortable and professional is essential to motivating students.

http://math.coe.uga.edu/TME/Issues/v03n2/Eggleton.pdf

# How to Motivate Adult Learners

- Focus on short-term goals
- Emphasize that all progress matters
- Let them know it's okay to get it wrong
- Make it relevant to their lives
- Tap into their intrinsic motivation

https://www.seedsofliteracy.org/5-ways-to-motivate-adult-learners/

$$z = b^{\ln n}$$

$$\ln z = \ln b^{\ln n}$$

$$\ln z = (\ln n)(\ln b)$$

$$z = b^{\ln n}$$

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$$\ln z = (\ln n)(\ln b)$$

$$z = e^{(\ln n)(\ln b)}$$

$$z = n^{\ln b}$$

$$z = \frac{1}{n^{-\ln b}}$$

$$z = b^{\ln n}$$

$$\ln z = \ln b^{\ln n}$$

$$\ln z = (\ln n)(\ln b)$$

$$z = e^{(\ln n)(\ln b)}$$

$$z = n^{\ln b}$$

$$z = \frac{1}{n^{-\ln b}}$$

$$-\ln b > 1$$

$$\ln b < -1$$

$$b < e^{-1}$$

$$b < \frac{1}{e}$$

$$\lim_{x \to 3} \frac{2x^2 - 2ax + x - a - 1}{x^2 - 2x - 3}$$

If 20-7a ≠0, the limit will not exist

Plugging in 3 for x, we get

$$\frac{2(3)^2 - 2a(3) + 3 - a - 1}{(3)^2 - 2(3) - 3} =$$

$$\frac{18 - 6a + 3 - a - 1}{9 - 6 - 3}$$

$$\frac{20 - 7a}{0}$$

Let a = 20/7 and then show that the limit exists

$$\lim_{x\to 3}\frac{2x^2-2ax+x-a-1}{x^2-2x-3}=$$

$$\lim_{x\to 3}\frac{2x^2-2\left(\frac{20}{7}\right)x+x-\frac{20}{7}-1}{x^2-2x-3}=$$

$$\lim_{x\to 3}\frac{2x^2-\left(\frac{40}{7}\right)x+x-\frac{27}{7}}{x^2-2x-3}=$$

$$\lim_{x\to 3}\frac{2x^2-\left(\frac{33}{7}\right)x-\frac{27}{7}}{x^2-2x-3}=$$

$$\lim_{x\to 3}\frac{14x^2-33x-27}{7\left(x^2-2x-3\right)}=$$

$$\lim_{x\to 3}\frac{(14x+9)(x-3)}{7(x-3)(x+1)}=$$

$$\lim_{x\to 3}\frac{(14x+9)}{7(x+1)}=$$

$$\frac{14(3)+9}{7(3+1)}=\frac{42+9}{7\cdot 4}=\frac{51}{28}$$

# Thank you

- Robert Cappetta
- cappetta@uic.edu
- Slides available at

  sites.google.com/site/RWCAMATYC/files
- Look for AMATYC2017.pdf