BUILDING COMMON SENSE IN CALCULUS THROUGH EXPERIENCE

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GOOD TEACHING FACTORS

CLASSROOM INTERACTIONS THAT ACKNOWLEDGE STUDENTS

ENCOURAGING AND AVAILABLE INSTRUCTOR

FAIR ASSESSMENT
A GOOD TEACHING OR A GOOD LEARNING ENVIRONMENT

- Student-faculty contact
- Cooperation among students
- Active learning
- Prompt feedback
- Time on task
- High expectations
- Respect for diverse talents and ways of learning

(Mesa, Burn and White in [1] ) Chickering & Gamson (1991) identify these principles via surveys of large numbers of student in various types of post-secondary institutions and a variety of disciplines.
EXPECT SUCCESS!

- Having High aspirations for students learning
- Setting clear expectations for student performance
- Establishing standards for holding students accountable

(Mesa, Burn and White in [1]) (see e.g., Hassel & Laurey, 2005; Tagg, 2003).
BUILDING COMMON SENSE AND CONFIDENCE

- Understand
- Analyze
- Generalize
  
  “Be Wise, Generalize”

- Strategy
  
  (To be developed or use an existing one)
SOME HELP FOR THE STUDENT TO BUILD COMMON SENSE

Strategies

• Rules of Differentiation
• Techniques of Integration
• Limits
• Sequences and Series
• Related Rates Problems
• Optimization
RULES OF DIFFERENTIATION
IRS-STRATEGY

Identify  Replace  Simplify
Special Functions

\( (\mathbf{n}^n)' = n\mathbf{n}^{n-1}\mathbf{n} \)
\( (\sqrt{\mathbf{n}})' = \frac{\mathbf{n}'}{2\sqrt{\mathbf{n}}} \)

\( (e^\mathbf{n})' = \mathbf{n}'e^\mathbf{n} \)

\( (\ln \mathbf{n})' = \frac{\mathbf{n}'}{\mathbf{n}} \)

Basic Rules

\( (cf(x))' = cf'(x) \)
\( (A + B)' = A' + B' \)

\( (AB)' = A'B + AB' \)

\( \left( \frac{A}{B} \right)' = \frac{BA' - AB'}{B^2} \)

Trigonometric Functions

\( (\sin \mathbf{n})' = \mathbf{n}' \cos \mathbf{n} \)
\( Trig^n \mathbf{n} = (Trig \mathbf{n})^n \)
\( (\tan \mathbf{n})' = \mathbf{n}' \sec^2 \mathbf{n} \)
\( (\sec \mathbf{n})' = \mathbf{n}' \sec \mathbf{n} \tan \mathbf{n} \)
\( (\cos \mathbf{n})' = -\mathbf{n}' \sin \mathbf{n} \)
\( (\cot \mathbf{n})' = -\mathbf{n}' \csc^2 \mathbf{n} \)
\( (\csc \mathbf{n})' = -\mathbf{n}' \csc \mathbf{n} \cot \mathbf{n} \)
TECHNIQUES OF INTEGRATION

• U-Substitution \( \int g'(x)f'[g(x)]dx \)
  
  Here you take \( u = g(x) \) the inside function,

  • Compute \( \frac{du}{dx} = g'(x) \), Solve for \( dx \) \( \Rightarrow dx = \frac{du}{g'(x)} \)

  • Substitute \( u \) and \( dx \) in the original integral, simplify and integrate.

\[
\int g'(x)f'[g(x)]dx = \int g'(x)f'(u) \frac{du}{g'(x)} = \int f'(u)du = f(u) + C
\]

• Go back to the substitution. \( \int g'(x)f'[g(x)]dx = f[g(x)] + C \)
3. Trigonometric Substitution

This technique is good for integrals that contain one of the following expressions:

- $\sqrt{x^2 + a^2}$ with corresponding triangle

\[
\begin{align*}
\frac{x}{a} = ? & \quad \frac{\sqrt{x^2 + a^2}}{a} = ? \\
x = ? & \quad \Rightarrow \quad dx = ? \quad \text{and} \quad \sqrt{x^2 + a^2} = ?
\end{align*}
\]

- $\sqrt{x^2 - a^2}$ with corresponding triangle

\[
\begin{align*}
\frac{x}{a} = ? & \quad \frac{\sqrt{x^2 - a^2}}{a} = ? \\
x = ? & \quad \Rightarrow \quad dx = ? \quad \text{and} \quad \sqrt{x^2 - a^2} = ?
\end{align*}
\]

- $\sqrt{a^2 - x^2}$ with corresponding triangle

\[
\begin{align*}
\frac{x}{a} = ? & \quad \frac{\sqrt{a^2 - x^2}}{a} = ? \\
x = ? & \quad \Rightarrow \quad dx = ? \quad \text{and} \quad \sqrt{a^2 - x^2} = ?
\end{align*}
\]
TECHNIQUES OF INTEGRATION

• Trigonometric Substitution

\[ \int \frac{dx}{x^3\sqrt{x^2 + a^2}} \]
LIMITS

Computing Limits When $x$ approaches infinity

This chart shows the order of domination of some special functions.

For example, to compute limits when $x$ approaches infinite:

$$\lim_{x \to \infty} \left( \frac{f(x)}{g(x)} \right) = \begin{cases} 0, & \text{if } g \text{ dominates } f \\ \infty, & \text{if } f \text{ dominates } g \end{cases}$$
SEQUENCES AND SERIES
Modified Limit comparison test (for Series)

Given $\sum_{n=k}^{\infty} a_n$

If $a_n \sim b_n$ then
the series $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=k}^{\infty} b_n$

Both converge or both diverge!!
\[
\sum_{n=1}^{\infty} \frac{n + 2^n}{n^2 2^n}
\]

\[
\sum_{n=1}^{\infty} \frac{3^n}{2^n + 3^n}
\]

\[
\sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}
\]
RELATED RATES PROBLEMS

1. Identify the given quantities and rates.
2. Identify the equation (Eq.) relating the quantities in the problem.
3. Use this (Eq.) and the given values.
4. Differentiate this equation. (Eq.)'.
5. Replace all the values and rates in the DE.
6. Solve for the missing rate.
7. Translate the answers in terms of the question in the problem...

This will be the differential equation. (Eq.)'=DE, (This Eq. relate the rates).
OPTIMIZATION STRATEGY

1. **Identify** 
   Identify the goal function GF (Usually a 2-variable function).

2. **Identify** 
   Identify the restriction Equation RE (Usually a 2-variable equation).

3. **Solve** 
   Solve for one of the variables in the RE.

4. **Substitute** 
   Substitute this variable in the GF (To get a new 1-variable new GF).

5. **Differentiate** 
   Differentiate the new GF.

6. **Solve** 
   Solve the equation \((\text{new GF})' = 0\).

7. **Check** 
   Check the answers for Max/min.

8. **Translate** 
   Translate this answers in terms of the problem or question.
THANK YOU!!

AMATYC 2018
Quiz Basic Rules of Differentiation

Special Functions

\((\boxed{n})' = n \boxed{n-1}\)  \((\sqrt{\boxed{\text{}}})' = \frac{\boxed{'}}{2\sqrt{\boxed{\text{}}}}\)

\((e^{\boxed{\text{}}})' = \boxed{'}e^{\boxed{\text{}}}\)

\((\ln \boxed{\text{}})' = \frac{\boxed{}}{\boxed{\text{}}}\)

Basic Rules

\((cf(x))' = cf'(x)\)  \((A + B)' = A' + B'\)

\((AB)' = A'B + AB'\)

\((A)' = \frac{BA' - AB'}{B^2}\)

Trigonometric Functions

\((\sin \boxed{\text{}})' = \boxed{'}\cos \boxed{\text{}}\)  \(\text{Trig}^n \boxed{\text{}} = (\text{Trig} \boxed{\text{}})^n\)

\((\tan \boxed{\text{}})' = \boxed{'}\sec^2 \boxed{\text{}}\)

\((\sec \boxed{\text{}})' = \boxed{'}\sec \boxed{\text{}}\tan \boxed{\text{}}\)

\((\cos \boxed{\text{}})' = -\boxed{'}\sin \boxed{\text{}}\)

\((\cot \boxed{\text{}})' = -\boxed{'}\csc^2 \boxed{\text{}}\)

\((\csc \boxed{\text{}})' = -\boxed{'}\csc \boxed{\text{}}\cot \boxed{\text{}}\)
Inverse Trigonometric Functions

Here we assume that \( \bullet \) is in the domain of the respective inverse trigonometric function

\[
\left( \sin^{-1} \bullet \right)' = \frac{\bullet'}{\sqrt{1 - \bullet^2}}
\]

\[
\left( \cos^{-1} \bullet \right)' = -\frac{\bullet'}{\sqrt{1 - \bullet^2}}
\]

\[
\left( \tan^{-1} \bullet \right)' = \frac{\bullet'}{1 + \bullet^2}
\]

\[
\left( \cot^{-1} \bullet \right) = -\frac{\bullet'}{1 + \bullet^2}
\]

\[
\left( \sec^{-1} \bullet \right)' = \frac{\bullet'}{|\bullet|\sqrt{|\bullet|^2 - 1}}
\]

\[
\left( \csc^{-1} \bullet \right)' = -\frac{\bullet'}{|\bullet|\sqrt{|\bullet|^2 - 1}}
\]
Special Functions

\[(\boldsymbol{n}^n)' = n\boldsymbol{n}^{n-1}\boldsymbol{n}'\]  \[ (\sqrt{\boldsymbol{n}})' = \frac{\boldsymbol{n}'}{2\sqrt{\boldsymbol{n}}} \]

\[(e^{\boldsymbol{n}})' = \boldsymbol{n}'e^{\boldsymbol{n}}\]

\[(\ln \boldsymbol{n})' = \frac{\boldsymbol{n}'}{\boldsymbol{n}}\]

Basic Rules

\[(cf(x))' = cf'(x)\]  \[ (A + B)' = A' + B' \]

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Trigonometric Functions

\[(\sin \boldsymbol{n})' = \boldsymbol{n}' \cos \boldsymbol{n}\]  \[ Trig^n\boldsymbol{n} = (Trig\boldsymbol{n})^n \]

\[(\tan \boldsymbol{n})' = \boldsymbol{n}' \sec^2 \boldsymbol{n}\]

\[(\sec \boldsymbol{n})' = \boldsymbol{n}' \sec \boldsymbol{n} \tan \boldsymbol{n}\]

\[(\cos \boldsymbol{n})' = -\boldsymbol{n}' \sin \boldsymbol{n}\]

\[(\cot \boldsymbol{n})' = -\boldsymbol{n}' \csc^2 \boldsymbol{n}\]

\[(\csc \boldsymbol{n})' = -\boldsymbol{n}' \csc \boldsymbol{n} \cot \boldsymbol{n}\]
Inverse Trigonometric Functions

Here we assume that □ is in the domain of the respective inverse trigonometric function

\[
\begin{align*}
\left( \sin^{-1} \square \right)' &= \frac{\square'}{\sqrt{1 - \square^2}} \\
\left( \cos^{-1} \square \right)' &= -\frac{\square'}{\sqrt{1 - \square^2}} \\
\left( \tan^{-1} \square \right)' &= \frac{\square'}{1 + \square^2} \\
\left( \cot^{-1} \square \right) &= -\frac{\square'}{1 + \square^2} \\
\left( \sec^{-1} \square \right)' &= \frac{\square'}{|\square|\sqrt{\square^2 - 1}} \\
\left( \csc^{-1} \square \right)' &= -\frac{\square'}{|\square|\sqrt{\square^2 - 1}}
\end{align*}
\]
Quiz Basic Rules of Differentiation

Name ________________________________

Special Functions

\((\text{n}^n)') = \quad (\sqrt{\text{n}})' = \)

\((e^{\text{n}})' = \)

\((\ln \text{n})' = \)

Basic Rules

\((cf (x)') = \quad (A + B)' = \)

\((AB)' = \)

\((A)' = \quad (\frac{A}{B})' = \)

Trigonometric Functions

\((\sin \text{n})' = \quad Trig^n \text{n} = \)

\((\tan \text{n})' = \)

\((\sec \text{n})' = \)

\((\cos \text{n})' = \)

\((\cot \text{n})' = \)

\((\csc \text{n})' = \)
Inverse Trigonometric Functions

Here we assume that $\bullet$ is in the domain of the respective inverse trigonometric function

\[
\begin{align*}
\left(\sin^{-1} \bullet\right)' &= \\
\left(\cos^{-1} \bullet\right)' &= \\
\left(\tan^{-1} \bullet\right)' &= \\
\left(\cot^{-1} \bullet\right) &= \\
\left(\sec^{-1} \bullet\right)' &= \\
\left(\csc^{-1} \bullet\right)' &= 
\end{align*}
\]
Computing Limits When \( x \) approaches infinity

This chart shows the order of domination of some special functions

For example, to compute limits when \( x \) approaches infinite:

\[
\lim_{x \to \infty} \left( \frac{f(x)}{g(x)} \right) = \begin{cases} 
0, & \text{if } g \text{ dominates } f \\
\infty, & \text{if } f \text{ dominates } g
\end{cases}
\]
for \( n \) sufficiently large. Then taking \( a_n = (\ln n)/n^{3/2} \) and \( b_n = 1/n^{5/4} \), we have

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\ln n}{n^{1/4}} = \frac{1}{n^{1/4}} = \lim_{n \to \infty} (1/4)n^{-3/4} = 0.
\]

(l’Hôpital’s Rule)

Since \( \sum b_n = \sum (1/n^{5/4}) \) is a \( p \)-series with \( p > 1 \), it converges, so \( \sum a_n \) converges by Part 2 of the Limit Comparison Test.

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### Exercises 10.4

#### Comparison Test

In Exercises 1–8, use the Comparison Test to determine if each series converges or diverges.

1. \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 30} \)
2. \( \sum_{n=1}^{\infty} \frac{n-1}{n^4 + 2} \)
3. \( \sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{n} - 1} \)
4. \( \sum_{n=1}^{\infty} \frac{n + 2}{n^2 - n} \)
5. \( \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{7/4}} \)
6. \( \sum_{n=1}^{\infty} \frac{1}{n^5} \)
7. \( \sum_{n=1}^{\infty} \frac{\sqrt{n + 4}}{n^4 + 4} \)
8. \( \sum_{n=1}^{\infty} \frac{\sqrt{n} + 1}{n^2 + 3} \)

#### Limit Comparison Test

In Exercises 9–16, use the Limit Comparison Test to determine if each series converges or diverges.

9. \( \sum_{n=1}^{\infty} \frac{n - 2}{n^3 - n^2 + 3} \)
(Hint: Limit Comparison with \( \sum_{n=1}^{\infty} (1/n^2) \))
10. \( \sum_{n=1}^{\infty} \frac{n + 1}{\sqrt{n^2 + 2}} \)
(Hint: Limit Comparison with \( \sum_{n=1}^{\infty} (1/\sqrt{n}) \))
11. \( \sum_{n=1}^{\infty} \frac{n(n+1)}{n^2 + 1(n-1)} \)
12. \( \sum_{n=1}^{\infty} \frac{2^n}{3 + 4^n} \)
13. \( \sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n^4} \)
14. \( \sum_{n=1}^{\infty} \left( \frac{2n + 3}{5n + 4} \right)^n \)
15. \( \sum_{n=1}^{\infty} \frac{1}{\ln n} \)
(Hint: Limit Comparison with \( \sum_{n=1}^{\infty} (1/n) \))
16. \( \sum_{n=1}^{\infty} \ln \left( 1 + \frac{1}{n^2} \right) \)
(Hint: Limit Comparison with \( \sum_{n=1}^{\infty} (1/n^2) \))

#### Determining Convergence or Divergence

Which of the series in Exercises 17–54 converge, and which diverge? Use any method, and give reasons for your answers.

17. \( \sum_{n=1}^{\infty} \frac{1}{2 \sqrt{n} + \sqrt{n}} \)
18. \( \sum_{n=1}^{\infty} \frac{3}{n + \sqrt{n}} \)
19. \( \sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n} \)
20. \( \sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2} \)
21. \( \sum_{n=1}^{\infty} \frac{2n}{3^n - 1} \)
22. \( \sum_{n=1}^{\infty} \frac{n + 1}{n^2 \sqrt{n}} \)
23. \( \sum_{n=1}^{\infty} \frac{10n + 1}{n(n+1)(n+2)} \)
24. \( \sum_{n=1}^{\infty} \frac{5n^4 - 3n}{n^2(n-2)(n^2 + 1)} \)
25. \( \sum_{n=1}^{\infty} \frac{n}{(3n + 1)^4} \)
26. \( \sum_{n=1}^{\infty} \frac{1}{n^3 + 2} \)
27. \( \sum_{n=1}^{\infty} \frac{1}{\ln (1/n)} \)
28. \( \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3} \)
29. \( \sum_{n=1}^{\infty} \frac{1}{n^3 \ln n} \)
30. \( \sum_{n=1}^{\infty} (\ln n)^2 \)
31. \( \sum_{n=1}^{\infty} \frac{1}{n + 1 + \ln n} \)
32. \( \sum_{n=1}^{\infty} \ln (n + 1) \n^2 \)
33. \( \sum_{n=1}^{\infty} \frac{1}{\ln (n + 1)} \n^2 \)
34. \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1} \)
35. \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 2n^2} \)
36. \( \sum_{n=1}^{\infty} \frac{n + 2n}{n^2 + 1} \)
37. \( \sum_{n=1}^{\infty} \frac{3n + 1}{3^n} \)
38. \( \sum_{n=1}^{\infty} \frac{3^n - 1}{3^n} \)
39. \( \sum_{n=1}^{\infty} \frac{n + 1}{n^2 + 3n + 1} \)
40. \( \sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n} \)
41. \( \sum_{n=1}^{\infty} \frac{2^n - n}{n^2} \)
42. \( \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n} e^n} \)
43. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)
(Hint: First show that \((1/n!) \leq (1/(n(n-1))) \) for \( n \geq 2 \))
44. \( \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!} \)
45. \( \sum_{n=1}^{\infty} \frac{\sin 1/n}{n} \)
46. \( \sum_{n=1}^{\infty} \frac{\tan 1/n}{n} \)
47. \( \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2 + 1} \)
48. \( \sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^2 + 1} \)
49. \( \sum_{n=1}^{\infty} \frac{\coth n}{n^2} \)
50. \( \sum_{n=1}^{\infty} \frac{\tanh n}{n^2} \)
51. \( \sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{n}} \)
52. \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} \)
53. \( \sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + \cdots + n} \)
54. \( \sum_{n=1}^{\infty} \frac{1}{1 + 2^2 + 3^2 + \cdots + n^2} \)

#### Theory and Examples

55. Prove (a) Part 2 and (b) Part 3 of the Limit Comparison Test.
56. If \( \sum_{n=1}^{\infty} a_n \) is a convergent series of nonnegative numbers, can anything be said about \( \sum_{n=1}^{\infty} (a_n/n) ? \) Explain.
57. Suppose that \( a_n > 0 \) and \( b_n > 0 \) for \( n \geq N \) (\( N \) an integer). If \( \lim_{n \to \infty} (a_n/b_n) = \infty \) and \( \sum b_n \) converges, can anything be said about \( \sum b_n \)? Give reasons for your answer.
58. Prove that if \( \sum a_n \) is a convergent series of nonnegative terms, then \( \sum a_n^2 \) converges.
Techniques of Integration

1. **U-substitution:** When we have an integral of the form

\[ \int g'(x)f'[g(x)]dx \]

Here you take

- \( u = g(x) \) the inside function
- Compute \( \frac{du}{dx} = g'(x) \)
- Solve for \( dx \), \( dx = \frac{du}{g'(x)} \)
- Substitute \( u \) and \( dx \) in the original integral, simplify (in this step all the x’s go away) and integrate.
- Go back to the substitution. \( \int g'(x)f'[g(x)]dx = f[g(x)] + C \)

2. **Integration by parts:**

This is good for the following type of integrals

\[ \int (\text{polynomials}) \left[ \begin{array}{l} \text{Exponential} \\ \text{or} \\ \text{Trigonometric} \end{array} \right] dx \]

In this case you use the “table technique” namely:

<table>
<thead>
<tr>
<th>Derivative</th>
<th>integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>Polynomial</td>
</tr>
<tr>
<td>-</td>
<td></td>
</tr>
<tr>
<td>+</td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

The other possibility is for integrals of the form

\[ \int (\text{polynomials}) \left[ \begin{array}{l} \text{Logarithmic} \\ \text{or} \\ \text{Inverse Trig.} \end{array} \right] dx \]

In this case you use the formula of integration by parts

\[ uv - \int vdu \]

Then take

\[ u = \left( \begin{array}{l} \text{Logarithmic} \\ \text{or} \\ \text{Inverse Trig.} \end{array} \right) \quad \text{And} \quad dv = (\text{polynomial})dx \]
3. **Trigonometric Substitution**

This technique is good for integrals that contain one of the following expressions:

- **Soh, Cah, Toa**
  - Csc, Sec, Cot

- $\sqrt{x^2 + a^2}$ with corresponding triangle

  $\frac{x}{a} = ?$  \[ \frac{\sqrt{x^2 + a^2}}{a} = ? \]

  $x = ?$  \[ \Rightarrow \]  \( dx = ? \) and $\sqrt{x^2 + a^2} = ?$

- $\sqrt{x^2 - a^2}$ with corresponding triangle

  $\frac{x}{a} = ?$  \[ \frac{\sqrt{x^2 - a^2}}{a} = ? \]

  $x = ?$  \[ \Rightarrow \]  \( dx = ? \) and $\sqrt{x^2 - a^2} = ?$

- $\sqrt{a^2 - x^2}$ with corresponding triangle

  $\frac{x}{a} = ?$  \[ \frac{\sqrt{a^2 - x^2}}{a} = ? \]

  $x = ?$  \[ \Rightarrow \]  \( dx = ? \) and $\sqrt{a^2 - x^2} = ?$
4. Trigonometric Integrals

Products of Powers of Sines and Cosines

We begin with integrals of the form:

\[ \int \sin^m x \cos^n x \, dx, \]

where \( m \) and \( n \) are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to \( m \) and \( n \) being odd or even.

**Case 1** If \( m \) is odd, we write \( m \) as \( 2k + 1 \) and use the identity \( \sin^2 x = 1 - \cos^2 x \) to obtain

\[
\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}
\]

Then we combine the single \( \sin x \) with \( dx \) in the integral and set \( \sin x \, dx \) equal to \(-d(\cos x)\).

**Case 2** If \( m \) is even and \( n \) is odd in \( \int \sin^m x \cos^n x \, dx \), we write \( n \) as \( 2k + 1 \) and use the identity \( \cos^2 x = 1 - \sin^2 x \) to obtain

\[
\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.
\]

We then combine the single \( \cos x \) with \( dx \) and set \( \cos x \, dx \) equal to \( d(\sin x) \).

**Case 3** If both \( m \) and \( n \) are even in \( \int \sin^m x \cos^n x \, dx \), we substitute

\[
\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}
\]

to reduce the integrand to one in lower powers of \( \cos 2x \).
5. **Partial Fractions Decomposition**

This technique applies when the integrand is a rational function

\[
\frac{p(x)}{q(x)} = \frac{\text{polynomial}}{\text{polynomial}}
\]

Provided that the degree of the numerator \( p(x) \) is less than the degree of the numerator \( q(x) \). If this is not the case perform long division.

**Method of Partial Fractions \((f(x)/g(x)\) Proper)\)**

1. Let \( x - r \) be a linear factor of \( g(x) \). Suppose that \((x - r)^m\) is the highest power of \( x - r \) that divides \( g(x) \). Then, to this factor, assign the sum of the \( m \) partial fractions:

\[
\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.
\]

Do this for each distinct linear factor of \( g(x) \).

2. Let \( x^2 + px + q \) be an irreducible quadratic factor of \( g(x) \) so that \( x^2 + px + q \) has no real roots. Suppose that \((x^2 + px + q)^n\) is the highest power of this factor that divides \( g(x) \). Then, to this factor, assign the sum of the \( n \) partial fractions:

\[
\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.
\]

Do this for each distinct quadratic factor of \( g(x) \).

3. Set the original fraction \( f(x)/g(x) \) equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of \( x \).

4. Equate the coefficients of corresponding powers of \( x \) and solve the resulting equations for the undetermined coefficients.
Approximations

**The Trapezoidal Rule**
To approximate $\int_a^b f(x) \, dx$, use

$$T = \frac{\Delta x}{2} \left( y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n \right).$$

The $y$’s are the values of $f$ at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n - 1)\Delta x, x_n = b,$$

where $\Delta x = (b - a)/n$.

**Simpson’s Rule**
To approximate $\int_a^b f(x) \, dx$, use

$$S = \frac{\Delta x}{3} \left( y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n \right).$$

The $y$’s are the values of $f$ at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n - 1)\Delta x, x_n = b.$$

The number $n$ is even, and $\Delta x = (b - a)/n$. 
Sequences

**Theorem 1** Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of real numbers, and let \( A \) and \( B \) be real numbers. The following rules hold if \( \lim_{n \to \infty} a_n = A \) and \( \lim_{n \to \infty} b_n = B \).

1. **Sum Rule:** \( \lim_{n \to \infty} (a_n + b_n) = A + B \)
2. **Difference Rule:** \( \lim_{n \to \infty} (a_n - b_n) = A - B \)
3. **Constant Multiple Rule:** \( \lim_{n \to \infty} (k \cdot b_n) = k \cdot B \) (any number \( k \))
4. **Product Rule:** \( \lim_{n \to \infty} (a_n \cdot b_n) = A \cdot B \)
5. **Quotient Rule:** \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{A}{B} \) if \( B \neq 0 \)

**Theorem 2**—The Sandwich Theorem for Sequences Let \( \{a_n\} \), \( \{b_n\} \), and \( \{c_n\} \) be sequences of real numbers. If \( a_n \leq b_n \leq c_n \) holds for all \( n \) beyond some index \( N \), and if \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \), then \( \lim_{n \to \infty} b_n = L \) also.

**Theorem 4** Suppose that \( f(x) \) is a function defined for all \( x \geq n_0 \) and that \( \{a_n\} \) is a sequence of real numbers such that \( a_n = f(n) \) for \( n \geq n_0 \). Then

\[ \lim_{x \to \infty} f(x) = L \quad \Rightarrow \quad \lim_{n \to \infty} a_n = L. \]

**Theorem 5** The following six sequences converge to the limits listed below:

1. \( \lim_{n \to \infty} \frac{\ln n}{n} = 0 \)
2. \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \)
3. \( \lim_{n \to \infty} x^{1/n} = 1 \quad (x > 0) \)
4. \( \lim_{n \to \infty} x^n = 0 \quad (|x| < 1) \)
5. \( \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad \text{(any } x) \)
6. \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \quad \text{(any } x) \)

In Formulas (3) through (6), \( x \) remains fixed as \( n \to \infty \).
**Definition**  Integrals with infinite limits of integration are improper integrals of Type I.

1. If \( f(x) \) is continuous on \([a, \infty)\), then
   \[
   \int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx.
   \]

2. If \( f(x) \) is continuous on \((-\infty, b]\), then
   \[
   \int_{-\infty}^b f(x) \, dx = \lim_{a \to -\infty} \int_a^b f(x) \, dx.
   \]

3. If \( f(x) \) is continuous on \((-\infty, \infty)\), then
   \[
   \int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^c f(x) \, dx + \int_c^{\infty} f(x) \, dx,
   \]
   where \( c \) is any real number.

In each case, if the limit is finite we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.
DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at $a$ then
   
   $$\int_a^b f(x) \, dx = \lim_{c \to a^+} \int_c^b f(x) \, dx.$$ 

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at $b$, then
   
   $$\int_a^b f(x) \, dx = \lim_{c \to b^-} \int_a^c f(x) \, dx.$$ 

3. If $f(x)$ is discontinuous at $c$, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then
   
   $$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$ 

In each case, if the limit is finite we say the improper integral converges and that the limit is the value of the improper integral. If the limit does not exist, the integral diverges.
Infinite Series

**THEOREM 8** If \( \sum a_n = A \) and \( \sum b_n = B \) are convergent series, then

1. **Sum Rule:** \( \sum (a_n + b_n) = \sum a_n + \sum b_n = A + B \)
2. **Difference Rule:** \( \sum (a_n - b_n) = \sum a_n - \sum b_n = A - B \)
3. **Constant Multiple Rule:** \( \sum k a_n = k \sum a_n = kA \) (any number \( k \)).

Convergence Test

1. **The \( n \)th-Term Test:** Unless \( a_n \to 0 \), the series diverges.
2. **Geometric series:** \( \sum ar^n \) converges if \( |r| < 1 \); otherwise it diverges.
3. **\( p \)-series:** \( \sum 1/n^p \) converges if \( p > 1 \); otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison Test.
5. **Series with some negative terms:** Does \( \sum |a_n| \) converge? If yes, so does \( \sum a_n \) since absolute convergence implies convergence.
6. **Alternating series:** \( \sum a_n \) converges if the series satisfies the conditions of the Alternating Series Test.

The \( n \)th-Term Test for Divergence

\[
\sum_{n=1}^{\infty} a_n \text{ diverges if } \lim_{n \to \infty} a_n \text{ fails to exist or is different from zero.}
\]

If \( |r| < 1 \), the geometric series \( a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \) converges to \( a/(1 - r) \):

\[
\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \quad |r| < 1.
\]

If \( |r| \geq 1 \), the series diverges.

**THEOREM 9—The Integral Test** Let \( \{a_n\} \) be a sequence of positive terms. Suppose that \( a_n = f(n) \), where \( f \) is a continuous, positive, decreasing function of \( x \) for all \( x \geq N \) (\( N \) a positive integer). Then the series \( \sum_{n=N}^{\infty} a_n \) and the integral \( \int_{N}^{\infty} f(x) \, dx \) both converge or both diverge.
**Theorem 11—Limit Comparison Test**  
Suppose that \( a_n > 0 \) and \( b_n > 0 \) for all \( n \geq N \) (\( N \) an integer).

1. If \( \lim_{{n \to \infty}} \frac{a_n}{b_n} = c > 0 \), then \( \sum a_n \) and \( \sum b_n \) both converge or both diverge.

2. If \( \lim_{{n \to \infty}} \frac{a_n}{b_n} = 0 \) and \( \sum b_n \) converges, then \( \sum a_n \) converges.

3. If \( \lim_{{n \to \infty}} \frac{a_n}{b_n} = \infty \) and \( \sum b_n \) diverges, then \( \sum a_n \) diverges.

**Theorem 12—The Ratio Test**  
Let \( \sum a_n \) be a series with positive terms and suppose that

\[
\lim_{{n \to \infty}} \frac{a_{n+1}}{a_n} = \rho.
\]

Then (a) the series converges if \( \rho < 1 \), (b) the series diverges if \( \rho > 1 \) or \( \rho \) is infinite, (c) the test is inconclusive if \( \rho = 1 \).

**Theorem 13—The Root Test**  
Let \( \sum a_n \) be a series with \( a_n \geq 0 \) for \( n \geq N \), and suppose that

\[
\lim_{{n \to \infty}} \sqrt[n]{a_n} = \rho.
\]

Then (a) the series converges if \( \rho < 1 \), (b) the series diverges if \( \rho > 1 \) or \( \rho \) is infinite, (c) the test is inconclusive if \( \rho = 1 \).

**Theorem 14—The Alternating Series Test (Leibniz’s Test)**  
The series

\[
\sum_{{n=1}}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots
\]

converges if all three of the following conditions are satisfied:

1. The \( u_n \)'s are all positive.
2. The positive \( u_n \)'s are (eventually) nonincreasing: \( u_n \geq u_{n+1} \) for all \( n \geq N \), for some integer \( N \).
3. \( u_n \to 0 \).
**THEOREM 16**—The Absolute Convergence Test

If \( \sum_{n=1}^{\infty} |a_n| \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

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**Power Series**

**DEFINITIONS**

A power series about \( x = 0 \) is a series of the form

\[
\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \tag{1}
\]

A power series about \( x = a \) is a series of the form

\[
\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n + \cdots \tag{2}
\]

in which the center \( a \) and the coefficients \( c_0, c_1, c_2, \ldots, c_n, \ldots \) are constants.

---

**How to Test a Power Series for Convergence**

1. Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely. Ordinarily, this is an open interval

\[
|x - a| < R \quad \text{or} \quad a - R < x < a + R.
\]

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is \( a - R < x < a + R \), the series diverges for \( |x - a| > R \) (it does not even converge conditionally) because the \( n \)th term does not approach zero for those values of \( x \).

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**THEOREM 20**

If \( \sum_{n=0}^{\infty} a_n x^n \) converges absolutely for \( |x| < R \), then \( \sum_{n=0}^{\infty} a_n (f(x))^n \) converges absolutely for any continuous function \( f \) on \( |f(x)| < R \).
THEOREM 21—The Term-by-Term Differentiation Theorem

If \( \sum c_n(x - a)^n \) has radius of convergence \( R > 0 \), it defines a function

\[
f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \quad \text{on the interval} \quad a - R < x < a + R.
\]

This function \( f \) has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

\[
f'(x) = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1},
\]

\[
f''(x) = \sum_{n=2}^{\infty} n(n - 1)c_n(x - a)^{n-2},
\]

and so on. Each of these derived series converges at every point of the interval \( a - R < x < a + R \).

THEOREM 22—The Term-by-Term Integration Theorem

Suppose that

\[
f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n
\]

converges for \( a - R < x < a + R \) \( (R > 0) \). Then

\[
\sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}
\]

converges for \( a - R < x < a + R \) and

\[
\int f(x) \, dx = \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} + C
\]

for \( a - R < x < a + R \).
Taylor and McLaurin Series

**DEFINITIONS** Let \( f \) be a function with derivatives of all orders throughout some interval containing \( a \) as an interior point. Then the Taylor series generated by \( f \) at \( x = a \) is

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n + \cdots.
\]

The Maclaurin series generated by \( f \) is

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots,
\]

the Taylor series generated by \( f \) at \( x = 0 \).

Taylor Polynomial

**DEFINITION** Let \( f \) be a function with derivatives of order \( k \) for \( k = 1, 2, \ldots, N \) in some interval containing \( a \) as an interior point. Then for any integer \( n \) from 0 through \( N \), the Taylor polynomial of order \( n \) generated by \( f \) at \( x = a \) is the polynomial

\[
P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots + \frac{f^{(k)}(a)}{k!} (x - a)^k + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n.
\]

**TABLE 10.1** Frequently used Taylor series

\[
\begin{align*}
\frac{1}{1-x} & = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \\
\frac{1}{1+x} & = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1 \\
e^x & = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty \\
\sin x & = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty \\
\cos x & = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad |x| < \infty \\
\ln (1 + x) & = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad -1 < x \leq 1 \\
\tan^{-1} x & = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1} \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| \leq 1
\end{align*}
\]