Complex Numbers:
From “Impossibility” to Necessity

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The nineteenth century psychologist, Hermann Ebbinghaus (1850-1908), observed that “psychology has a long past but only a short history.” He meant to convey the fact that while psychological phenomena had been observed for millennia, the systematic study of them was young. A similar comment could be made about complex numbers. Although first encountered in antiquity, their importance was not recognized until the sixteenth century, and they were poorly understood until the nineteenth. We would like to trace this development from its origin to its importance in mathematics today.

Let us begin with Heron of Alexandria (10-70), best known for the formula that calculates the area of a triangle in terms of its sides. His work *Stereometria* includes his attempt to find the height of a frustum of a pyramid with a square base. The height h is given by the formula $h=\sqrt{c^2-2\left(\frac{a-b}{2}\right)^2}$, where c is the slant height of the frustum and a and b are the lengths of the sides of the bases. Heron set a = 28, b = 4, and c =15 to obtain $h=\sqrt{-63}$, which is the first known appearance of the square root of a negative number in the history of mathematics. However, he did not conclude that such a frustum is impossible. Although he was on the verge of discovering complex numbers, in his notes he replaced -63 with 63.

Diophantus (c. 214-296), also of Alexandria, encountered a similar situation. One of the questions he posed in his *Arithmetrica* was to find the sides of a right triangle, the area and perimeter of which are 7 and 12 respectively. His formulation of the problem led to a quadratic equation with the discriminant of -167. He stated that there was no solution and did not analyze the matter further.

For many centuries no effort was exerted to understand square roots of negative numbers. The technique of completing the square, developed by the ancient Babylonians around 2000 B.C., was used to solve quadratic equations; whenever a negative discriminant was obtained, mathematicians echoed Diophantus and concluded that the equation had no solution. For example, in the ninth century Madhavira, who wrote the first Hindu mathematics textbook, reaffirmed the view that such square roots do not exist.

The motivation to understand complex numbers arose from the need to solve cubic equations rather than quadratic ones. Such equations appeared in the work of Diophantus, Omar Khayyam (1050-1123), the Persian poet and mathematician known to us as the author of the *Rubaiyat*, and Fibonacci (1170-1250). Attempts were made to develop a universal method of calculating their roots, and Khayyam devised a geometric technique to solve any cubic having a positive solution. However, he did not believe that a general algebraic approach using a cubic formula similar to the quadratic formula could be found.

In 1494, the Italian mathematician, Luca Pacioli (1445-1514), published the first printed European work on algebra. Entitled the *Summa*, it also contained material on arithmetic and Euclidean geometry and provided a systematic discussion of double-entry bookkeeping, a contribution that earned Pacioli recognition as the father of accounting. He became famous throughout Italy as a teacher of mathematics; a picture of him tutoring a student in geometry hangs in the Naples Museum.

Pacioli closed the *Summa* by claiming that an algebraic solution of the cubic equation was impossible. In 1501 and 1502, he lectured at the University of Bologna, where one of his colleagues was Scipione del Ferro (1465-1526). It is not known whether del Ferro was motivated by Pacioli, but a few years later he solved algebraically the equation $x^3 + px = q, \text{ where } p \text{ and } q \text{ are positive}$. The equation is known as the depressed cubic since the quadratic term is missing. Although del Ferro's method is also correct for negative coefficients, he required p and q to be positive because at that time in Europe, negative numbers were poorly understood.

The general form of a cubic equation, of course, is $ax^3 + bx^2 + cx + d = 0$. Late in the fourteenth century, two anonymous Florentine manuscripts had shown that making the substitution $x = y - \frac{b}{3a}$ in this equation eliminates the quadratic term and converts it to a depressed cubic. Thus, by solving the latter, del Ferro was not sacrificing generality. At first glance the motivation for this substitution seems obscure, but a closer look reveals a similarity to the process of completing the square.

Del Ferro concealed his achievement because mathematicians of his era frequently tried to advance their careers by competing against each other in problem solving contests. Only late in his life did he communicate his result to a student, Antonio Maria Fior. Word of the discovery began to spread, and in 1545, Jerome Cardano (1501-1576), another Italian mathematician and physician, showed how to solve the general cubic in his book, *Ars Magna*. Cardano was an unscrupulous character, who admitted that the solution was not original with him. Indeed, he solved the equation only after receiving a hint from Niccolò Tartaglia (ca. 1500-1557), yet another Italian mathematician. Publishing the result betrayed his promise to Tartaglia that he would wait until the latter had done so. However, he subsequently learned that del Ferro had done the original work, and he concluded that his promise to Tartaglia was no longer binding. Cardano's action infuriated Tartaglia, and a bitter feud ensued between them.

As a child, Tartaglia suffered a severe sabre wound that affected his ability to speak. Although his last name was Fontana, he was referred to as Tartaglia, that is, the stammerer, a nickname that he eventually adopted. Because of his speech impediment, he found it difficult to earn a living, and he hoped his solution of the cubic would secure his financial future. It is not clear whether he solved the equation independently or received help, but because Cardano published the result first, he never received the recognition he sought.

Let us look at Cardano's solution of the depressed cubic as written above. The method could be called completing the cube because it is based on the identity, $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$, which can be rewritten as $(a-b)^3 + 3ab(a-b) = a^3 - b^3$. If there are values of a and b such that 3ab = p and $a^3 - b^3 = q$, then $(a-b)^3 + p(a-b) = q$, and consequently x = a – b is a solution. The system of equations, 3ab = p and $a^3 - b^3 = q$ can be solved to obtain $a = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ *and* $b = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$. Finding a requires solving a sixth degree equation that is quadratic in a^3. It follows that $x = a - b = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$. It is worth noting that the technique of solving equations that are quadratic in form is necessary to obtain this answer. Students often struggle with this method, but Cardano's solution demonstrates its historical importance.

Cardano also solved equations of the form $x^3 = px + q$, *where p and q are positive*. By using the identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)$ he arrived at $x = a + b = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$. He recognized that the common radicand of the square root terms in this solution could be negative. For example, the equation $x^3 = 15x + 4$ yields $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$. Cardano did not reject complex number solutions outright, but because he did not know how to interpret them, he declared that they were "sophistical" and "as subtle as they were useless."

Rafael Bombelli (1526-1573) took a different approach. It is easy to see that four is a solution of $x^3 = 15x + 4$. He observed that in the solution a + b of this equation, the two cube root radicands differ only by a sign. We would say, of course, that they are complex conjugates. He conjectured and was able to prove that a and b themselves can be written as conjugates, specifically, that $a = 2 + \sqrt{-1}$ *and* $b = 2 - \sqrt{-1}$. He then added a and b as if they were real numbers to obtain the solution of 4. Although he did not justify this step, his work suggested that real numbers could be written in complex number form and thus that complex number solutions of equations could not be ignored.

In response to Bombelli's work, mathematicians began to search for a geometric justification of complex numbers. René Descartes (1596-1650) knew how to construct a line segment, the length of which is the square root of a given segment, but his method failed when he tried to construct the square root of a negative number. Because he equated mathematical existence with geometric construction, he concluded that it was impossible to establish the reality of complex numbers.

Descartes did accept the Fundamental Theorem of Algebra, and he recognized the need to use complex numbers to implement it. Albert Girard (1595-1632) was the first to state that a polynomial equation has as many solutions as its degree, and he recognized complex solutions, as well as roots of multiplicity greater than one. Although Descartes agreed, he introduced the term "imaginary" into the vocabulary of mathematics by saying that "for any equation one can imagine (as many roots as its degree would suggest), but in many cases no quantity exists which corresponds to what one imagines."

Descartes stated the Factor Theorem in his work on the theory of equations, and he used it to justify the Fundamental Theorem of Algebra. He explained that if r is a root of the polynomial f(x) of degree n, then x – r is a factor of it. Dividing f(x) by this factor produces a polynomial of degree n – 1, and repetition of this process ultimately yields n roots. His description did not constitute a proof because he assumed the existence of a root, whereas the purpose of the theorem is to establish this fact. Nevertheless, his explanation provided an intuitive understanding of it. Gauss gave the first satisfactory proof in 1799.

John Wallis (1616-1703) disagreed with Descartes' conclusion that complex numbers cannot be represented geometrically. Girard had improved understanding of negative numbers by placing negative solutions of equations on a number line to the left of a point selected to represent zero, just as positive solutions were placed to the right of zero. Wallis generalized Girard's approach to say that all negative and positive numbers could be associated with points to the left and right of zero respectively. He then suggested that complex numbers are two dimensional by saying that square roots of negative numbers correspond to points on a line perpendicular to a real number line. He did not, however, pursue his idea to define the complex plane.

Neither Isaac Newton (1642-1727) nor Gottfried Leibniz (1646-1716), both younger contemporaries of Wallis, attached much significance to complex numbers.

Like Descartes, they held that a complex number answer to a question indicated that the problem had no solution. Leibniz referred to the square root of a negative number as "an amphibian between being and not-being."

During the eighteenth century mathematicians gradually became more comfortable using complex numbers in intermediate stages of solving problems because their answers could be confirmed. Nevertheless, there was uncertainty about whether their methods could be justified. Leonhard Euler's work exemplified this approach. In 1770 he wrote:

> Because all conceivable numbers are greater than zero or less than zero or equal to zero, then it is clear that the square roots of negative numbers cannot be included among the possible numbers. Consequently we must say that these are impossible numbers. And this circumstance leads us to the concept of such numbers, which by their nature are impossible, and ordinarily are called imaginary or fancied numbers, because they exist only in the imagination.

Clearly, Euler mistakenly thought that to be legitimate, complex numbers had to obey the trichotomy law. He also made errors in his attempt to work with them. Because $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for real numbers, he assumed that this relationship also holds in the complex case. Like Descartes, Newton, and Leibniz, he interpreted a complex number answer to mean that a problem has no solution. For instance, if we wish to find two numbers whose sum is 12 and product is 40, we would obtain answers of 6 + 2i and 6 – 2i. Euler said that there is no solution.

Although Euler did not understand the nature of complex numbers, he used them to make fundamental contributions to mathematics. One of his most important was his definition of $e^{ix} = \cos x + i \sin x$. Known today as Euler's Formula, he motivated it by replacing x by ix in the Maclaurin series for e^x and then separating the result into its real and imaginary parts. Today we use Euler's Formula, for example, to write complex numbers in polar form and to represent solutions of a higher order linear homogeneous differential equation with constant coefficients if the auxiliary equation has complex roots.

For x = π, Euler's Formula says that $e^{\pi i} = \cos \pi + i \sin \pi = -1$, which can be rewritten, of course, as $e^{\pi i} + 1 = 0$. The numbers 0, 1, π, e, and i are five of the most important numbers in mathematics. Euler's Formula, which was called "the most remarkable formula in mathematics" by the physicist, Richard Feynman, highlights a fundamental relationship connecting them.

In addition to the many advances in mathematics for which Euler was responsible, he also contributed to mathematical notation by introducing the symbols i and π that we use today.

Only at the end of the eighteenth century was there an improvement in the understanding of the nature of complex numbers. In 1799, Caspar Wessel (1745-1818), a Norwegian surveyor and cartographer, published a paper in a Danish journal entitled "On the Analytical Representation of Direction." His focus was not on complex numbers as such but on describing both length and direction in two dimensions algebraically by means of a single expression. To do so, he viewed line segments as vectors, which he added by means of the standard parallelogram law and multiplied as we do complex numbers today. Wessel's work allowed mathematicians to visualize the complex number a + bi as the terminal point (a,b) of a vector whose initial point is the origin and thereby removed much of the mystery surrounding the concept.

The journal in which Wessel published his paper was not widely read, and his seminal contribution received little attention. Not until 1897, almost a century later, was his work rediscovered and republished,

Jean-Robert Argand (1768-1822), a Swiss bookkeeper, also displayed complex numbers as vectors, but he took a different approach. He observed that negative numbers form an extension of the positive numbers obtained by considering direction as well as magnitude, and he asked whether there is an extension of the real numbers that would produce the complex numbers. He noted that if a number x is multiplied by -1, the location of -x on a number line represents a rotation of 180 degrees from the position of x. By writing $-1=\sqrt{-1}\sqrt{-1},$ he concluded that multiplication by $i=\sqrt{-1}$ twice achieves the same result. He then surmised that multiplying by it only once corresponds to a rotation of ninety degrees and generates the imaginary number ix, located on an axis perpendicular to a real axis, as Wallis had suggested. In general, he associated the complex number a + bi with the vector of magnitude $\sqrt{a^2+b^2}$ and direction given by the angle in standard position determined by the real part a and the imaginary part b. Although Argand's work received more attention than Wessel's, it too went largely unnoticed.

The geometric representation of complex numbers reached its final form in the work of Carl Gauss (1777-1855), who dispensed with the vectorial approach of Wessel and Argand and associated the complex number a + bi with the point (a,b). His insight showed that complex numbers arithmetize the plane just as real numbers arithmetize the line. In fact, he introduced the term "complex number" to emphasize the two-dimensional nature of the concept.

Concerning his theory of complex numbers, Gauss wrote:

> That this subject has hitherto been considered from the wrong point of view and surrounded by a mysterious obscurity is to be attributed largely to an ill-adapted notation. If for instance, +1, -1, and $\sqrt{-1}$ had been called direct,

> inverse, and lateral units, instead of positive, negative, and imaginary (or even impossible) such an obscurity would have been out of the question.

Gauss' view of complex numbers was basic to his work on the Fundamental Theorem of Algebra. Following Descartes' discussion of the theorem, Newton, D'Alembert, Euler, and Lagrange unsuccessfully attempted to prove it. In 1799, Gauss gave the first acceptable proof by working with the real and imaginary parts of complex-valued functions. Although his argument was not fully acceptable by modern standards, it convinced mathematicians that the theorem was true and inaugurated a greater emphasis on rigor during the nineteenth century.

The final step in the effort to explain complex numbers was taken by the Irish mathematician, William Rowan Hamilton (1805-1865), who wanted to devise a strictly arithmetic definition, free of any geometric context. He observed that expressions of the form a + bi are not sums in the sense that a + b is, and he replaced them by ordered pairs of real numbers, subject to the standard rules of complex number arithmetic. In this formulation the symbol i does not appear and becomes the ordered pair (0,1), which, when multiplied by itself, yields (-1,0), the two-dimensional equivalent of -1. Hamilton did acknowledge that writing (a, b) as a + bi was useful for computational purposes. In private correspondence Gauss stated that he also had the idea of defining complex numbers in the same manner. However, because of his tendency to publish slowly, Hamilton was the first mathematician to state this definition publicly.

Hamilton wanted to extend the idea of a complex number to three dimensions in order to represent forces that do not lie in the same plane. After some fifteen years of effort, he realized that he had to abandon the commutative property of multiplication in order to do so. He introduced the concept of a quaternion, an expression of the form a + bi +cj +dk, where a is a scalar and the symbols i, j, and k have a function similar to the complex number i. Specifically, the square of each has the value of -1, as does the product ijk. The lack of commutativity results from the fact that ij = k, jk = i, and ki =j, but ji =-k, kj = -i, and ik =-j. Hamilton's identification of these relationships came to him in a flash of insight as he was walking across a bridge in his home city of Dublin, Ireland, and to commemorate his discovery, he carved the inscription $i^2 = j^2 = k^2 = ijk = -1$ into the bridge with his knife.

Hamilton's hope that quaternions would become the basis of mathematical physics was not realized, but his work still led to a significant advance. By dispensing with the scalar term of a quaternion and defining i, j, and k to be the unit vectors along the x-, y-, and z-axes respectively, Josiah Willard Gibbs (1839-1903) and Oliver Heaviside (1850-1925) independently laid the foundation for three-dimensional vector analysis. They defined the dot and cross products of two vectors, and Gibbs introduced the standard notation for each. As we know, the cross product is noncommutative, and unlike the case with numbers, it can be zero even though neither vector is. This latter property is also true of the dot product although it is commutative.

Hamilton's introduction of noncommutative multiplication led to the investigation of new algebraic structures. Arthur Cayley (1821-1895) made fundamental contributions to matrix theory by formulating many of its basic concepts, including the addition of matrices, the zero matrix, and the identity matrix. Of special importance was his definition of matrix multiplication, which, of course, is not commutative. He observed that a system of linear equations in the same number of unknowns can be expressed in matrix form and solved by using the inverse of the coefficient matrix; he also noted that this method is not universal: If the determinant of the coefficient matrix is zero, then the matrix is not invertible.

Let us now look at some applications and consequences of the development of complex numbers. The design of alternating current circuits in electrical engineering uses the vectorial properties of the concept to represent both magnitude and phase shift. Mechanical engineering uses complex numbers in order to assess the stability of systems.

Classical physics needs complex numbers to solve differential equations, and they appear in the wave equation in quantum mechanics.

We have noted that Hamilton's attempt to generalize the concept of a complex number led to quaternions and the completely unanticipated development of three-dimensional vector analysis. Cardano's work also had an unexpected consequence. His secretary, Ludovico Ferarri, solved the quartic equation by a method similar to the one published by Cardano to solve the cubic. His method consisted of making the substitution $x = y - \frac{b}{4a}$ in the general quartic to eliminate the cubic term and then completing the square to find a solution.

It was natural to try to solve the general quintic, but all of the efforts failed. Lagrange was one of those who was unsuccessful, and he conjectured that no solution was possible. This was later established by Niels Abel (1802-1829), and Evaríste Galois (1811-1832) found conditions under which a solution is possible. The work of Abel and Galois shifted the focus of algebra from computation to the study of mathematical structure and led to such concepts as groups, rings, and fields. This development was unconnected to the investigation of complex numbers, but it was another outgrowth of the attempt to solve higher degree polynomial equations.

Almost two millennia passed between Heron's encounter with complex numbers and Hamilton's rigorous definition of the concept. Gauss captured the challenge of this effort when he observed that "the metaphysics of the square root of -1 is difficult." His realization that the complex numbers arithmetize the plane just as the real numbers arithmetize the line prepared the way for the next phase in the history of complex numbers: the creation of the subject of complex analysis, established in the nineteenth century primarily by Augustin Cauchy (1791-1857), Bernhard Riemann (!826-1866), and Karl Weierstrass (1815-1895). The Riemann hypothesis, one of the outstanding unsolved problems in mathematics today, belongs to this field. We began by saying that complex numbers have a long past but only a short history. Based on their contributions to mathematics thus far, let us conclude by anticipating that they will continue to have a significant impact on the subject in the future.

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