Looking High and Low to Integrate Rational Functions

By:

Diane Koenig

Rock Valley College

Rock Valley College

- Rock Valley College is a two-year college in Rockford, IL. It is located in northern Illinois near the center of the top of the state, approximately 90 miles northwest of Chicago

Rational integrand with monomial denominator

- $\int \frac{5x^4 - 3x^3 + 4x - 1}{4x} dx$

The integration:

$$\int \left(\frac{5x^4}{4x} - \frac{3x^3}{4x} + \frac{4x}{4x} - \frac{1}{4x} \right) dx$$

$$\int \left(\frac{5}{4}x^3 - \frac{3}{4}x^2 + 1 - \frac{1}{4}x^{-1} \right) dx$$

$$\frac{5}{16}x^4 - \frac{1}{4}x^3 + x - \frac{1}{4}ln|x| + c$$

Look alikes:

- $\int \frac{2x+6}{x^2+6x+7} dx$
- $\int \frac{2x+4}{x^2+6x+9} dx$

The integration:

- $\int \frac{2x+6}{x^2+6x+7} dx$
- $u = x^2 + 6x + 7$
- $du = (2x + 6)dx$
- $\int \frac{1}{u} du$
- $\ln|u| + c$
- $\ln|x^2 + 6x + 7| + c$

- $\int \frac{2x+4}{x^2+6x+9} dx$
- $\int \frac{2x+4}{(x+3)^2} dx$
- $\frac{A}{x+3} + \frac{B}{(x+3)^2}$
- $\int \left(\frac{2}{x+3} + \frac{-2}{(x+3)^2}\right) dx$
- $2\ln|x + 3| + \frac{2}{x+3} + c$

Look alikes:

- $\int \frac{2x+6}{x^2+6x+7}\,dx$
- $\int \frac{2x+6}{(x^2+6x+7)^3}\,dx$

The integration:

- $\int \frac{2x+6}{x^2+6x+7} dx$
- $u = x^2 + 6x + 7$
- $du = (2x + 6)dx$
- $\int \frac{1}{u} du$
- $\ln|u| + c$
- $\ln|x^2 + 6x + 7| + c$

- $\int \frac{2x+6}{(x^2+6x+7)^3} dx$
- $u = x^2 + 6x + 7$
- $du = (2x + 6)dx$
- $\int \frac{1}{u^3} du$
- $\int u^{-3} du$
- $-\frac{1}{2(x^2+6x+7)^2} + c$

Look alikes:

$$\int \frac{2x^3}{x^4 + 25} dx$$

$$\int \frac{2x}{x^4 + 25} dx$$

The integration:

$$\int \frac{2x^3}{x^4 + 25} dx$$

- $\int \frac{2x^3}{x^4+25} dx$
- $u = x^4 + 25$
- $du = 4x^3 dx$
- $\frac{1}{2} \int \frac{1}{u} du$
- $\frac{1}{2} \ln|u| + c$
- $\frac{1}{2} \ln(x^4 + 25) + c$

$$\int \frac{2x}{x^4 + 25} dx$$

- $\int \frac{2x}{x^4+25} dx$
- $u^2 = x^4$ $\qquad a^2 = 25$
- $u = x^2$ $\qquad a = 5$
- $du = 2xdx$
- $\int \frac{1}{u^2+a^2} du$
- $\frac{1}{a} \tan^{-1} \frac{u}{a} + c$
- $\frac{1}{5} \tan^{-1} \frac{x^2}{5} + c$

Strategies:

Compare the numerator to the denominator:

1) If the numerator degree is the same or larger than the denominator, perform the division to write the rational expression in

$$quotient + \frac{remainder}{divisor}$$

$$\int \frac{x^3 + 5x^2 - 7x - 9}{x^2 + x - 1} dx$$

$$\int \left(x + 4 + \frac{-10x - 5}{x^2 + x - 1} \right) dx$$

$$\frac{1}{2}x^2 + 4x - 5\ln|x^2 + x - 1| + c$$

2) If the degree of the numerator is lower than the degree of the denominator by one degree, try:

A) a u-substitution using the entire denominator.

$$\int \frac{10x+5}{x^2+x-1} dx \qquad u = x^2 + x - 1$$

$$du = (2x + 1)dx$$

B) If the u-substitution is not exactly a fit, is it a constant multiple?

$$\int \frac{5(2x+1)}{x^2+x+1} dx$$

$$5 \int \frac{(2x+1)}{x^2+x+1} dx$$

$$5 \ln|u| + c$$

$$5 \ln|x^2 + x - 1| + c$$

C) Numerator degree less than the denominator by more than one degree try factoring the denominator and use a u-substitution

$$\int \frac{6}{x^2+6x+9}dx$$

$\int \frac{6}{(x+3)^2}dx$ $u = x+3$ $6\int u^{-2}du$

$du = dx$ $-6u^{-1}+c$

$$-\frac{6}{x+3}+c$$

D) factoring the denominator as a sum of two squares to a form an integrand whose integral is inverse tangent

$$\int \frac{6}{x^2+6x+13}dx$$

$$\int \frac{6}{(x+3)^2+4}dx$$

$$3\tan^{-1}\frac{x+3}{2}+c$$

E) factoring the denominator and use the process of partial fraction.

$$\int \frac{2x + 4}{x^2 + 10x + 25} dx$$

$$\int \frac{2x + 4}{(x + 5)^2} dx$$

$$\int \frac{A}{x + 5} + \frac{B}{(x + 5)^2} dx$$

$$\int \frac{2}{x + 5} + \frac{-6}{(x + 5)^2} dx$$

$$2 \ln|x + 5| + \frac{6}{x + 5} + c$$

- While introducing the integration of rational functions that require a specific technique at different times in the curriculum is warranted, it is important to also revisit the techniques used on previous problems and compare the problems side by side. Have the students discuss what they notice, how they might be able to discern between the two.
- Often students do not stop and ponder the differing problem types and we should initiate a discussion of their comparisons.

Thank you for attending today
Contact information:

- Diane Koenig

- D.Koenig@rockvalleycollege.edu

Find the following indefinite integrals:

$$\int \frac{2x+6}{x^2+6x+7}\,dx$$

$$\int \frac{2x+4}{x^2+6x+9}\,dx$$

$$\int \frac{2x+6}{(x^2+6x+7)^3}\,dx$$

$$\int \frac{6}{x^2+6x+9}\,dx$$

$$\int \frac{6}{x^2+6x+10}\,dx$$

$$\int \frac{6}{x^2+6x+25}\,dx$$

$$\int \frac{5x^2-10x+5}{\frac{1}{3}x^3-x^2+x-9}\,dx$$

$$\int \frac{4x^2-61x+24}{x^3-13x^2+12x}\,dx$$

$$\int \frac{x}{x^4+16}\,dx$$

$$\int \frac{11x}{x^4+10x^2+25}\,dx$$

$$\int \frac{x^3+5x^2-7x+3}{x^2-1}\,dx$$

$$\int \frac{x+2}{x^2-4x}\,dx$$

$$\int \frac{7x-14}{x^2-4x}\,dx$$

$$\int \frac{6x}{x^4+6x^2+13}\,dx$$

$$\int \frac{-7x^2+14x-7}{\frac{1}{3}x^3-x^2+x-9}\,dx$$

$$\int \frac{x}{9x^4+7}\,dx$$

$$\int \frac{x^2-x}{x^2+x+1}\,dx$$

$$\int \frac{x^2+x+3}{x^4+6x^2+9}\,dx$$

$$\int \frac{x^3+5x^2-7x-9}{x^2+x-1}\,dx$$

$$\int \frac{x^4-3x^3+5x^2-x+9}{x-2}\,dx$$

$$\int \frac{5x^2}{x^6+11}\,dx$$

$$\int \frac{5x^3}{x^8+11}\,dx$$