SOME PROOFS TO ANALYZE

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1. A Proof by Induction

Theorem :

$$1 + 3 + 5 + ... + (2n - 1) = n^2$$

Proof

Basis Step: $1 = 1^2$

Inductive Step: Assume that the statement holds for some natural number k.

$$1 + 3 + 5 + ... + (2k - 1) = k^2 \qquad \text{(line 1)}$$

$$1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = k^2 + 2k + 1 \qquad \text{(line 2)}$$

$$1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = (k + 1)^2 \qquad \text{(line 3)}$$

Then $P(k + 1)$ is true, and by the principle of mathematical induction, $P(n)$ holds for all natural numbers n.

1.1. Explain the theorem in words

1.2. Show that it holds for $n = 5$

1.3. How did I get from line 1 to line 2? From 2 to 3?

1.4. What is $P(n)$?

2. A Proof by Strong Induction

Any natural number can be written in the form $n = 2^b \cdot r$, where $b \geq 0$ and r is an odd integer.

Proof : We break into 2 cases and use strong induction. Let $P(n)$ be the statement:

There exist integers b, r with $b \geq 0$ and r odd, such that $n = 2^b \cdot r$

Case 1 : n is odd. In that case, $n = 2^0 \cdot n$, and $P(n)$ is true.

Case 2 : n is even. Then there exists $k \in \mathbb{Z}$ such that $n = 2k$.

Then $k < n$ and by the inductive hypothesis, $P(k)$ must be true, so

$$k = 2^b \cdot r$$

Multiply both sides by two:

$$2k = 2 \cdot 2^b \cdot r$$
$$n = 2^{b+1} \cdot r$$

Then $P(n)$ is true.

2.1. Explain what the theorem means.

2.2. Confirm that it holds for the numbers 7, 8 and 12.

2.3. This is supposedly a proof by induction, but it's a bit informal. What's the base case?

2.4. What are we inducting ON? We aren't exactly going from k to $k + 1$ here.

2.5. The proof is broken into 2 cases. Do these 2 cases cover all possibilities?

2.6. Why must k be less than n?

3. Proof that the square root of three is irrational

Assume that $\sqrt{3} \in \mathbb{Q}$.

Then, by definition, there must be integers a, b such that $\sqrt{3} = \frac{a}{b}$. Assume without loss of generality that $\frac{a}{b}$ is in lowest terms. Then, let's do some algebra:

$$\sqrt{3} = \frac{a}{b}$$
$$3 = \left(\frac{a}{b}\right)^2$$
$$3b^2 = a^2$$

Then a^2 is divisible by 3, which means a must also be divisible by 3.

Let $a = 3k$. Then $a^2 = 9k^2$. But then we have:

$$3b^2 = 9k^2.$$
$$b^2 = 3k^2$$

Then b is divisible by 3, so a and b have a common factor.

But this contradicts our original assumption that $\frac{a}{b}$ is in lowest terms. Then $\sqrt{3}$ must be irrational.

3.1. What is the definition of an irrational number?

3.2. What proof method is used here?

3.3. Why is it safe to assume that $\frac{a}{b}$ is in lowest terms?

3.4. Why does a^2 divisible by 3 imply that a is divisible by 3?

3.5. How do you need to modify this to write a proof that $\sqrt{5}$ is irrational?

3.6. Challenge: Why does this proof method break down if you try to use it to prove that $\sqrt{4}$ is irrational? Note that I'm not asking why $\sqrt{4}$ is rational, I'm asking why this particular argument doesn't work.

4. Proof that there are infinitely many prime numbers

Proof : Assume that the largest prime number is n. Consider

$$Q_n = n! + 1$$

Then either Q_n is prime, or it has a prime factor. Call Q_n's smallest prime factor q_n.

But none of $\{2, 3...n\}$ can be factors of Q_n, so q_n must be greater than n. Which means we have found a prime greater than n, and there must be infinitely many primes.

4.1. What does $n!$ mean?

4.2. This is a proof by contradiction. Where does the contradiction happen?

4.3. Why can't Q_n be divisible by any of $\{2, 3...n\}$? (Hint: How is it defined?)

4.4. Assume that the only primes in the world are 2, 3, and 5. Use the proof method to generate another prime.

4.5. Will this proof generate all primes?