Spherical Excess in Antarctica

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We first consider the following definitions...
“The circle in which a plane through the center of a sphere intersects the sphere is called a great circle.”


“A spherical triangle consists of three arcs of great circles that form boundaries of a portion of the surface.”

“The law of cosines for sides. *The cosine of any side of a spherical triangle is equal to the product of the cosines of the two other sides increased by the product of the sines of the two other sides and the cosine of the angle included between them,* or in symbols

\[
\cos a = \cos b \cos c + \sin b \sin c \cos A.
\]

In example 97 on page 68 of Isaac Todhunter’s 1871 text, Spherical Trigonometry For The Use Of Colleges And Schools, With Numerous Examples we find that the area

of a triangle with vertices A, B and C is

\[(A + B + C - \pi)r^2\]

where \(A + B + C\) is called the spherical excess of the triangle. In my calculations, degrees are used.
Why calculate the areas of Antarctic territorial claims?
Answer:

This is a great way to introduce the concepts described in the previous slides since two of the three angles (A, B and C) are identical and the other angle is easily calculated.

However, the calculations shown in the document only represent approximations of the area since one side of each region does not lie on a great circle.
Now that we have considered the definitions we will be using let’s look at the Antarctic territories. Note the **Marie Byrd Land** is an unclaimed territory.
The 6th article of the Antarctic Treaty signed by representatives from twelve countries on December 1, 1959 states that the provisions of the treaty apply to the area south of the 60 degree south latitude.

For further information on the Antarctic Treaty you can visit the website: [https://ats.aq/e/antarctic treaty.html](https://ats.aq/e/antarctic treaty.html)
Seven countries (Argentina, Australia, Chile, France, New Zealand, Norway, and the United Kingdom) maintain territorial claims in Antarctica, but the United States and most other countries do not recognize those claims. While the United States maintains a basis to claim territory in Antarctica, it has not made a claim.”
At the Library of Congress website
https://www.loc.gov/item/91682864/
we find the following 1986 Central Intelligence Agency map:
What is interesting about the upper left-hand side of the map?
Answer:

The territorial claims of Argentina, Chile and the United Kingdom all overlap!
In the pdf document which will be available on the AMATYC conference site, the approximate areas were calculated using spherical law of cosines for sides and spherical excess assuming all sides lie on great circles. We can then compare those areas to the area derived by using a portion of the area of a spherical cap calculated using surface area formula for a spherical cap,

\[ A = 2\pi rh = 2\pi (3959 \text{ mi.}) \left( 3959 \text{ mi.} - \frac{\sqrt{3}}{2} (3959)\text{mi.} \right) \]

\[ \approx 13,193,904 \text{ sq. mi.} \]  See the website en.wikipedia.org/wiki/Spherical_cap for further details.
Let’s compare two areas, since a 6 degree angle is created at the south pole for Adelie Land, the area of that region using the area of the spherical cap is $13,193,904.27 \text{ sq. mi.} \left( \frac{6^\circ}{360^\circ} \right) \approx 219,898 \text{ sq. mi.} \; ;$

Using spherical trigonometry to approximate the area we get $219,574 \text{ sq. mi.}$.

The area using the spherical cap is 0.148% greater than the area using spherical trigonometry.
Let’s consider another comparison...
Let’s compare two areas, since a 91 degree angle is created at the south pole for Chilean Antarctic Territory, the area of that region using the area of the spherical cap is $13,193,904.27 \text{ sq. mi.} \left( \frac{37^\circ}{360^\circ} \right) \approx 1,356,040 \text{ sq. mi.} $;
Using spherical trigonometry to approximate the area we get $1,280,303 \text{ sq. mi.}$.

The area using the spherical cap is 5.916% greater than the area using spherical trigonometry.
Here are some suggested ideas for a group project in a trigonometry class:

In preparation for the project, select one of the territories, show the student how the area of a spherical triangle with the given vertices is calculated. Show how the area of the territory is calculated using the formula for the area of a spherical cap.

Ask the student to calculate the area of a spherical triangle using the vertices of a different Antarctic territory. Then ask the student to calculate the area of the territory using the area formula for a spherical cap. Ask them why the areas are different, and ask them if all sides of the territory lie on a great circle.
Area of a spherical triangle with the same vertices as Adélie Land

We now create a spherical triangle with sides of measure $90^\circ - 60^\circ = 30^\circ$, $90^\circ - 60^\circ = 30^\circ$, and $b$. The measure of the angle opposite side $b$ which we will call $\beta = 142^\circ - 136^\circ = 6^\circ$. 

South Pole
Now, we obtain the measure of side $b$.

$\cos b = \cos 30^\circ \cos 30^\circ + \sin 30^\circ \sin 30^\circ \cos 6^\circ$

$\cos b \approx 0.9986304738 \Rightarrow b \approx \cos^{-1}(0.9986304738) \Rightarrow b \approx 2.99897174^\circ$

Let's look once again at the our spherical triangle.
We proceed to find the measure of angles \( \alpha \) by once again utilizing the spherical law of cosines.

\[
\cos \alpha = \frac{\cos 30^\circ - \cos 30^\circ \cos 2.99897174^\circ}{\sin 30^\circ \sin 2.99897174^\circ}
\]

\[
\cos \alpha = 0.0453397938 \Rightarrow \alpha \approx \cos^{-1}(0.0453397938) \Rightarrow \alpha \approx 87.40133031^\circ.
\]

\[\therefore \text{Spherical Excess} = 2\alpha + \beta - 180^\circ \approx 2(87.40133031^\circ) + 6^\circ - 180^\circ \approx 0.8026606116^\circ.\]

Hence the area is

\[
A \approx \left(\frac{0.8026606116^\circ \cdot \pi}{180^\circ}\right)(3959)^2 \approx 219,574 \text{ square miles.}
\]
Area of a spherical triangle with the same vertices as Argentine Antarctica

We now create a spherical triangle with sides of measure \( 90^\circ - 60^\circ = 30^\circ \), \( 90^\circ - 60^\circ = 30^\circ \), and \( n \). The measure of the angle opposite side \( n \) is \( 74^\circ - 25^\circ = 49^\circ \).
Now, we obtain the measure of side $n$.

$$\cos n = \cos 30^\circ \cos 30^\circ + \sin 30^\circ \sin 30^\circ \cos 49^\circ$$

$$\cos n \approx 0.9140147572 \Rightarrow n \approx \cos^{-1}(0.9140147572) \Rightarrow n \approx 23.93380526^\circ$$

Let's look once again at our spherical triangle.

![Diagram of spherical triangle with $n \approx 23.93380526^\circ$ and angles labeled: $30^\circ$, $49^\circ$, and $30^\circ$. The triangle is centered around the South Pole.]
We proceed to find the measure of angles $\alpha$ by once again utilizing the spherical law of cosines.

\[
\cos \alpha = \frac{\cos 30^\circ - \cos 30^\circ \cos 23.93380526^\circ}{\sin 30^\circ \sin 23.93380526^\circ}.
\]

\[
\cos \alpha = 0.367113155 \Rightarrow \alpha \approx \cos^{-1}(0.367113155) \Rightarrow \alpha \approx 68.46231213^\circ.
\]

\[
\therefore \text{Spherical Excess} = 2\alpha + \gamma - 180^\circ \approx 2(68.46231213^\circ) + 49^\circ - 180^\circ \\
\approx 5.924624263^\circ.
\]

Hence the area is,

\[
A \approx \left(\frac{5.924624263^\circ \cdot \pi}{180^\circ}\right)(3959)^2 \approx 1,620,724 \text{ square miles}.
\]
Area of a spherical triangle with the same vertices as Australian Antarctic Territory

Australian Antarctic Territory consists of two separate regions separated by Adélie Land. We now calculate the area of one of these two regions. Create a spherical triangle with sides of measure $90^\circ - 60^\circ = 30^\circ$, $90^\circ - 60^\circ = 30^\circ$, and $b$. The measure of the angle opposite side $b$ which we will call $\beta = 136^\circ - 45^\circ = 91^\circ$. 

![Diagram of a spherical triangle with vertices at 60°S 45°E, South Pole, and 60°S 136°E, and sides of 30° and 30°]
Now, we obtain the measure of side $b$.

$$\cos b = \cos 30^\circ \cos 30^\circ + \sin 30^\circ \sin 30^\circ \cos 91^\circ$$

$$\cos b \approx 0.7456368984 \Rightarrow b \approx \cos^{-1}(0.7456368984) \Rightarrow b \approx 41.78616712^\circ$$

Let's look once again at our spherical triangle.
We proceed to find the measure of angles $\alpha$ by once again utilizing the spherical law of cosines.

$$\cos \alpha = \frac{\cos 30^\circ - \cos 30^\circ \cos 41.78616712^\circ}{\sin 30^\circ \sin 41.78616712^\circ}$$

$$\cos \alpha \approx 0.6611663266 \Rightarrow \alpha \approx \cos^{-1}(0.6611663266) \Rightarrow \alpha \approx 48.61111575^\circ$$

\[ \therefore \text{Spherical Excess} = 2\alpha + \beta - 180^\circ \approx 2(48.61111575^\circ) + 91^\circ - 180^\circ \approx 8.2223149^\circ. \]

Hence the area of the first region including is,

$$A_1 \approx \left(\frac{8.2223149^\circ \cdot \pi}{180^\circ}\right) (3959)^2 \approx 2,249,251.77 \text{ square miles.}$$
We now calculate the area of second of the second region. Create a spherical triangle with sides of measure $90^\circ - 60^\circ = 30^\circ$, $90^\circ - 60^\circ = 30^\circ$, and $b$. The measure of the angle opposite side $b$ which we will call $\beta = 160^\circ - 142^\circ = 18^\circ$. 

![Diagram of a spherical triangle with labels 60°S 142°E, 60°S 160°E, 30°, 30°, and 18° at the South Pole.]
Now, we obtain the measure of side $b$.

$$\cos b = \cos 30^\circ \cos 30^\circ + \sin 30^\circ \sin 30^\circ \cos 18^\circ$$

$$\cos b \approx 0.9877641291 \Rightarrow b \approx \cos^{-1}(0.9877641291) \Rightarrow b \approx 8.972199081^\circ$$

Let’s look once again at our spherical triangle.
We proceed to find the measure of angles $\alpha$ by once again utilizing the spherical law of cosines.

\[
\cos \alpha = \frac{\cos 30^\circ - \cos 30^\circ \cos 8.972199081^\circ}{\sin 30^\circ \sin 8.972199081^\circ}
\]

\[
\cos \alpha = 0.1358925494 \implies \alpha \approx \cos^{-1}(0.1358925494) \implies \alpha \approx 82.18976515^\circ
\]

\[
\therefore \text{Spherical Excess} = 2\alpha + \beta - 180^\circ \approx 2(82.18976515^\circ) + 18^\circ - 180^\circ \\
\approx 2.379530295^\circ.
\]

It follows that the area of second region (including all water and land),

\[
A_2 \approx \left(\frac{2.379530295^\circ \cdot \pi}{180^\circ}\right) (3959)^2 \approx 650,937.98 \text{ square miles.}
\]

**It follows that the total area of Australian Antarctic Territory,**

\[
A = A_1 + A_2 = 2,249,251.77 + 650,937.98 = 2,900,190 \text{ square miles}
\]
Area of a spherical triangle with the same vertices as British Antarctic Territory

We now create a spherical triangle with sides of measure $90^\circ - 60^\circ = 30^\circ$, $90^\circ - 60^\circ = 30^\circ$, and $b$. The measure of the angle opposite side $b$ which we will call $\beta = 80^\circ - 20^\circ = 60^\circ$. 

South Pole
Now, we obtain the measure of side \( b \).

\[
\cos b = \cos 30^{\circ}\cos 30^{\circ} + \sin 30^{\circ}\sin 30^{\circ}\cos 60^{\circ}
\]

\[
\cos b \approx 0.875 \Rightarrow b \approx \cos^{-1}(0.875) \Rightarrow b \approx 28.95502437^{\circ}
\]

Let's look once again at our spherical triangle.
We proceed to find the measure of angles $\alpha$ by once again utilizing the spherical law of cosines.

$$\cos \alpha = \frac{\cos 30^\circ - \cos 30^\circ \cos 28.95502437^\circ}{\sin 30^\circ \sin 28.95502437^\circ}$$

$$\cos \alpha = 0.4472135955 \Rightarrow \alpha \approx \cos^{-1}(0.4472135955) \Rightarrow \alpha \approx 63.43494882^\circ$$

\[\therefore \text{ Spherical Excess } = 2\alpha + \beta - 180^\circ \approx 2(63.43494882^\circ) + 60^\circ - 180^\circ \approx 6.86989764^\circ.\]

Hence the area is,

$$A \approx \left(\frac{6.86989764^\circ \cdot \pi}{180^\circ}\right)(3959)^2 \approx 1,879,311 \text{ square miles.}$$
Area of a spherical triangle with the same vertices as Chilean Antarctic Territory

We now create a spherical triangle with sides of measure $90^\circ - 60^\circ = 30^\circ$, $90^\circ - 60^\circ = 30^\circ$, and $b$. The measure of the angle opposite side $b$ which we will call $\beta = 90^\circ - 53^\circ = 37^\circ$. 

South Pole
Now, we obtain the measure of side $b$.

$$\cos b = \cos 30^\circ \cos 30^\circ + \sin 30^\circ \sin 30^\circ \cos 37^\circ$$

$$\cos b \approx 0.9496588775 \Rightarrow b \approx \cos^{-1}(0.9496588775) \Rightarrow b \approx 18.25736238^\circ$$

Let's look once again at our spherical triangle.

$b \approx 18.25736238^\circ$
We proceed to find the measure of angles $\alpha$ by once again utilizing the spherical law of cosines.

\[
\cos \alpha = \frac{\cos 30^\circ - \cos 30^\circ \cos 18.25736238^\circ}{\sin 30^\circ \sin 18.25736238^\circ}.
\]

\[
\cos \alpha = 0.2783189383 \Rightarrow \alpha \approx \cos^{-1}(0.2783189383) \Rightarrow \alpha \approx 73.84010071^\circ.
\]

\[\therefore \text{Spherical Excess} = 2\alpha + \beta - 180^\circ \approx 2(73.84010071^\circ) + 37^\circ - 180^\circ \approx 4.68020142^\circ.\]

Hence the area is,

\[A \approx \left(\frac{4.68020142^\circ \cdot \pi}{180^\circ}\right)(3959)^2 \approx 1,280,303 \text{ square miles}.\]
Area of a spherical triangle with the same vertices as Marie Byrd Land

We now create a spherical triangle with sides of measure $90^\circ - 60^\circ = 30^\circ$, $90^\circ - 60^\circ = 30^\circ$, and $b$. The measure of the angle opposite side $b$ which we will call $\beta = 158^\circ - 103^\circ 24' = 54^\circ 36'$. 
Now, we obtain the measure of side b. Note that $54^\circ 36' = 54.6^\circ$

$\cos b = \cos 30^\circ \cos 30^\circ + \sin 30^\circ \sin 30^\circ \cos 54.6^\circ$

$\cos b \approx 0.8948202931 \Rightarrow b \approx \cos^{-1}(0.8948202931) \Rightarrow b \approx 26.51464366^\circ$

Let's look once again at our spherical triangle.
We proceed to find the measure of angles \( \alpha \) by once again utilizing the spherical law of cosines.

\[
\cos \alpha = \frac{\cos 30^\circ - \cos 30^\circ \cos 26.51464366^\circ}{\sin 30^\circ \sin 26.51464366^\circ}
\]

\[
cos \alpha = 0.4080774454 \Rightarrow \alpha \approx \cos^{-1}(0.4080774454) \Rightarrow \alpha \approx 65.91587998^\circ
\]

\[
\therefore \text{Spherical Excess} = 2\alpha + \beta - 180^\circ \approx 2(65.91587998^\circ) + 54.6^\circ - 180^\circ
\]

\[
\approx 6.431759964^\circ.
\]

Hence the area is,

\[
A \approx \left( \frac{6.431759964^\circ \cdot \pi}{180^\circ} \right)(3959)^2 \approx 1,759,455 \text{ square miles}.
\]
Area of a spherical triangle with the same vertices as Queen Maud Land

We now create a spherical triangle with sides of measure $90^\circ - 60^\circ = 30^\circ$, $90^\circ - 60^\circ = 30^\circ$, and $b$. The measure of the angle opposite side $b$ which we will call $\beta = 20^\circ + 45^\circ = 65^\circ$. 

South Pole
Now, we obtain the measure of side $b$.

$$\cos b = \cos 30^\circ \cos 30^\circ + \sin 30^\circ \sin 30^\circ \cos 65^\circ$$

$$\cos b \approx 0.8556545654 \Rightarrow b \approx \cos^{-1}(0.8556545654) \Rightarrow b \approx 31.16787625^\circ$$

Let's look once again at the our spherical triangle.
We proceed to find the measure of angles \( \alpha \) by once again utilizing the spherical law of cosines.

\[
\cos \alpha = \frac{\cos 30^\circ - \cos 30^\circ \cos 31.16787625^\circ}{\sin 30^\circ \sin 31.16787625^\circ}
\]

\[
\cos \alpha = 0.4830739141 \Rightarrow \alpha \approx \cos^{-1}(0.4830739141) \Rightarrow \alpha \approx 61.1136426^\circ
\]

\[
\therefore \text{Spherical Excess} = 2\alpha + \beta - 180^\circ \approx 2(61.1136426^\circ) + 65^\circ - 180^\circ \\
\approx 7.2272852^\circ.
\]

Hence the area is,

\[
A \approx \left(\frac{7.2272852^\circ \cdot \pi}{180^\circ}\right)(3959)^2 \approx 1,977,077 \text{ square miles}.
\]
Area of a spherical triangle with the same vertices as Ross Dependency

We now create a spherical triangle with sides of measure $90^\circ - 60^\circ = 30^\circ$, $90^\circ - 60^\circ = 30^\circ$, and $b$. The measure of the angle opposite side $b$ which we will call $\beta = (180 - 160)^\circ + (180 - 150)^\circ = 50^\circ$. 

South Pole
Now, we obtain the measure of side $b$.

\[
\cos b = \cos 30^\circ \cos 30^\circ + \sin 30^\circ \sin 30^\circ \cos 50^\circ
\]

\[
\cos b \approx 0.9106969024 \Rightarrow b \approx \cos^{-1}(0.9106969024) \Rightarrow b \approx 24.39816338^\circ
\]

Let's look once again at the our spherical triangle.
We proceed to find the measure of angles $\alpha$ by once again utilizing the spherical law of cosines.

\[
\cos \alpha = \frac{\cos 30^\circ - \cos 30^\circ \cos 24.39816338^\circ}{\sin 30^\circ \sin 24.39816338^\circ}
\]

$\cos \alpha = 0.3744535818 \Rightarrow \alpha \approx \cos^{-1}(0.3744535818) \Rightarrow \alpha \approx 68.00945511^\circ$

$\therefore$ Spherical Excess $= 2\alpha + \beta - 180^\circ \approx 2(68.00945511^\circ) + 50^\circ - 180^\circ$

$\approx 6.01891022^\circ$.

Hence the area is,

\[
A \approx \left(\frac{6.01891022^\circ \cdot \pi}{180^\circ}\right)(3959)^2 \approx 1,646,517 \text{ square miles}.
\]