(Re)discovering Calculus in the Class Room

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• Position → Velocity
  • RATE of CHANGE of Composite Functions
  • RATE of CHANGE of Products of Functions

• Velocity → Position
  • Introduce ACCUMULATION
• Position → Velocity
• RATE of CHANGE of composite functions
First, students gathered data on the position of the end of an oscillating cantilevered beam...
...by digitizing the positions in LoggerPro...
...and entered those data into Excel.

In Excel they calculated average velocities between the time points...
...and copied the position and average velocity data into Desmos.
They then fit functions to the position data...

\[ f(x) = C \sin(a_1 x + b_1) \]

\[ f_1(x) = -C \cos(a_1 x) \]

\( C = 1.73 \)

\( a_1 = 55 \)

\( b_1 = -2.26 \)

\( x = b_1 \n x = -2.26 \)

\( d_1 = 1.78 \)
...and velocity data.

\[ y_{v1} = \frac{d}{dx} f(x) \{ a \leq x \leq \} \]

add \((x)\) to \(y_{v2}(x)\)

\[ y_{v2}(x) = C_1 \cos(a_1 x + \) \]

\[ y_{v3}(x) = a_1 C \cos(a_1 x + \) \]

\[ y_{v4}(x) = a_1 C \sin(a_1 x + \) \]

\[ b_2 = -0.53 \]

\[ C_1 = a_1 \cdot C \]
Drawing average slope lines...

$$
\int_{a}^{a + \left\lceil \frac{(x - a)N}{(b - a)} \right\rceil} f(x) \frac{\left\lceil \frac{(x - a)N}{(b - a)} \right\rceil}{N}
$$

- $N = 8$
- $b = 0.292$
- $a = 0$
...and increasing the number of slope lines.
Plotting the values of the slopes....
...shows the relationship between the slope of the position function and the velocity data.
Velocity amplitude = frequency \cdot position amplitude \rightarrow chain rule

\[ y(t) = C \sin(\omega t + \phi) + d \]

\[ u(t) = \omega t + \phi \]

\[ v(t) = \frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} \]

\[ v(t) = \omega C \cos(\omega t + \phi) \]
• Position → Velocity
  • RATE of CHANGE of composite functions
  • RATE of CHANGE Products of Functions
Rate of change of products of functions

• Initially the damping was ignored

• Now we add an exponential function to model the damping

• And introduce the product rule to find the damped velocity function
Rate of change of products of functions

\[ y(t) = Ce^{kt} \sin(\omega t + \phi) + d \]

\[ v(t) = \omega Ce^{kt} \cos(\omega t + \phi) + kCe^{kt} \sin(\omega t + \phi) \]
• Position → Velocity
  • RATE of CHANGE Products of Functions

• Velocity → Position
  • Introduce ACCUMULATION
...students summed the area under the velocity function to plot position points.
Radioactive Decay Cascade

\[
\frac{dR}{dt} = -\lambda_R \cdot R
\]

\[
\frac{dA}{dt} = -\lambda_A \cdot A + \lambda_R \cdot R
\]
Most fluids (liquids or gases) are *viscous*, meaning that they tend to "stick" to moving surfaces. The diagram shows a fluid between two plates that are separated by a distance \( b \). The upper plate is moving toward the right with constant velocity \( U \), and the lower plate is stationary. So the velocity of the fluid is 0 at the lower plate, and increases to \( U \) at the upper plate. The arrows in the diagram show the velocity vectors at various heights \( y \) from the bottom plate (the *velocity profile*) for a simple case.

a. If the velocity \( u \) at any height varies linearly with \( y \) as shown, write an equation for the function \( u = f(y) \).

b. Based on your answer to Part (a), find \( \frac{du}{dy} \), and explain what it means.

adapted from Munson, *Fundamentals of Fluid Mechanics*
Polypropylene

Stress (psi)

Strain (in/in)
Cardiac Work
Part of the energy used by your heart during its contraction consists of the *pumping work* of the left ventricle, which sends blood through the aorta to the rest of the body.

The figure below shows a *work loop diagram* for one contraction of a ventricle.

- Describe what appears to be happening during the work loop pictured on the graph. Explain what is occurring during each of the four phases of the cycle. Also explain how you know whether the direction of the work loop is clockwise or counter-clockwise.

- By measuring the coordinates of various points on the graph, accumulate information that allows you to compute the work done during the work loop. Explain your methodology, and state results in SI units.

adapted from Ethier & Simmons *Introductory Biomechanics*
More Accumulation
More Accumulation

1. Find the area of the ends of the dam.

2. Find the volume of concrete in the dam.
More Accumulation

1. Find the area of an octagonal slice.
2. Find the volume contained within the dome.
Domed hotel proposal for Quincy quarry gets a serious look

June 2, 1990

The Boston Globe (Boston, MA)
More Accumulation

1. Find the total volume of space in the quarry.

2. Find the above ground volume within the dome.

3. Find the dome's surface area
Thanks!
• Image references

• www.Radford.com
• Giordano, Weir, & Finney: *Calculus for Engineers and Scientists*
• *The Boston Globe*
Oscillating Motion - Part 1

Assignment Overview

This assignment is similar to the one where we gathered position data for a falling object, but this time we’ll study the oscillation of a spaghetti noodle. The spaghetti noodle will be clamped at one end to form a cantilever. Wikipedia defines “cantilever” as:

*A rigid structural element, such as a beam or a plate, anchored at only one end to a (usually vertical) support from which it is protruding.*

Cantilevers are used frequently in civil structures such as bridges and buildings (particularly overhanging structures such as balconies) and in mechanical devices (cantilevered springs, diving boards.)

You’ll collect position data on the cantilevered spaghetti noodle from a slow-motion video using Logger-Pro and import these data into Desmos. Then you’ll brainstorm what kind of function will best model this oscillatory motion and what parameters will be necessary. It should be noted that the mathematical model we will be obtaining here is very idealized: it assumes that the mass of the beam (noodle) is concentrated at the moving end and that the amplitude of the oscillation is small compared to the beam length.

The stress on a cantilevered beam with a static load may be calculated from the geometry of the beam, properties of the beam material and the beam deflection. However an analysis of a beam with a dynamic load must consider the oscillatory nature of the motion that results from the dynamic load.

Learning objectives for this assignment

<table>
<thead>
<tr>
<th>Learning objectives</th>
<th>Associated Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Become acquainted with terms involving loads or forces on cantilevered beams.</td>
<td>Part 1, step 2; perhaps elsewhere.</td>
</tr>
<tr>
<td>2. Become acquainted with terms involving oscillating motion, what functions can model oscillating motion and what parameters are involved in such a model.</td>
<td>Part 3</td>
</tr>
<tr>
<td>3. Model the oscillatory motion of an idealized undamped cantilever.</td>
<td>Part 3</td>
</tr>
</tbody>
</table>

Steps to complete the assignment

Part 1 – Vocabulary

1. What is a static load? What is a dynamic load?

2. View the video of the cantilevered spaghetti noodle in LoggerPro. Is there a static or dynamic load on the noodle? What causes the noodle to oscillate?
Part 2 – Gathering the data and entering them into Excel

3. Create a data set of the position of the end of the cantilever noodle vs. time in LoggerPro. To better visualize the end of the noodle a magenta post-it note is attached to the noodle’s end. It will be easier to trace the motion if you pick one of the corners of the post-it to track. Place the origin at the starting location of the corner of the post-it note you’ll be tracking. The video frame rate is 240 frames per second (fps). You’ll need to adjust this in LoggerPro to get the correct time points. To do this, select the movie window so that it’s highlighted, then, in the Options menu, select ”Movie Options...”. Check the ”Override frame rate...” box and input 240.

4. Collect data for 2 or 3 complete cycles of oscillation and enter the position and time data into Excel. Advance the video until the noodle is completely free from the finger before acquiring data.

5. Make a column that displays the time interval between points.

6. Create another new column of ”reset” time that begins at time zero (so that the release of the noodle begins at time 0) by subtracting the first time value from each entry in the time column.

7. Copy the reset time and the position data into a table in the [Desmos template]. Place your table under the folder Position Data. Tables in Desmos will only accommodate 50 entries, so you’ll have to split your data into multiple tables.
7. What type of function does the data seem to be following?
   (Initially we're going to ignore that the peaks seem to be getting less high and the valleys getting less deep with time. For this first modeling exercise pretend that the peaks are of equal height and the valleys of equal depth.)

Plot the function you’ve selected in Desmos. Locate the function in the folder *Undamped Position* in the location indicated below:

\[
f(x) = \text{place your function here} \{a \leq x \leq b\}
\]

Set the constants \(a\) and \(b\) (under the "Segments" folder) to the first and last time values in your position data. These will limit the domain of your plots.

8. How does your function look compared to the data? Chances are it doesn’t agree well. Don’t let that daunt you just yet. There are probably some parameters that you can tweak to improve the agreement between your chosen function and your data. What parameters can you think of to add (and then tweak?) If you’re not sure about this, think about in what ways your function and your data differ. Is your function too high or too short (amplitude), too “stretched” or not “stretched” enough (frequency)? Does it need to be offset in the position (equilibrium position) or time (phase angle) direction? Each of these is a parameter that you can include in your function and chances are you’ll need to adjust all four of them. Add these as constants in Desmos so they can be adjusted using “sliders.”

9. Once you’ve added the four parameters to your function (and their corresponding sliders) adjust their values until your function is close to overlying your data. (Note that your function will overshoot the peaks and valleys after the first peak - we’ll correct that a bit later.)

10. Save your Desmos file to your Desmos account - we’ll be using it later.
Oscillating Motion - Part 2

Assignment Overview

In this continuation of the previous assignment you’ll calculate the velocity of the cantilevered noodle at different points in time using average velocities and then find a function for the instantaneous velocity using limits. Try to recall the formula for average velocity. Check if you’re right by consulting one of the earlier worksheets.

After you’ve plotted an approximation of velocity using piecewise linear average velocity, you’ll use a limit expression to determine a function for the instantaneous velocity.

Your position function should look something like this:

\[ f(t) = C \sin(\omega t + \phi) + d_1 \]

Where:

- \( C \) = amplitude
- \( \omega \) = angular frequency
- \( \phi \) = phase angle
- \( d_1 \) = equilibrium position

Learning objectives for this assignment

<table>
<thead>
<tr>
<th>Learning objectives</th>
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</tr>
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<tbody>
<tr>
<td>1. Understand the parameters associated with a sine (or cosine) function</td>
<td>Part 1</td>
</tr>
<tr>
<td>2. Find average velocities of an oscillating cantilever using a piecewise linear function.</td>
<td>Part 2</td>
</tr>
<tr>
<td>3. Find a function for the velocity of an oscillating cantilever empirically.</td>
<td>Part 3</td>
</tr>
<tr>
<td>4. Model the oscillatory motion of a damped cantilever.</td>
<td>Part 4</td>
</tr>
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</table>

Steps to complete the assignment

Part 1 – Exploring the parameters of the sine function

1. In part 1 of this worksheet you fit a sine function to your position data by manipulating the values of the function parameters by hand and “eyeballing” the results. Now we will take a somewhat more analytical approach. We’ll start with \( d_1 \) the offset or equilibrium position.
The equilibrium position will lie midway between the peaks (function maxima) and valleys (function minima). Since we’re ignoring the gradual decrease in the heights and depths of the peaks and valleys we’ll work with the just the first peak and the first valley. The equilibrium position, \( d_1 \) will be the average value of the values of the first peak and the first valley. Calculate \( d_1 \).

2. \( C \), the amplitude is the height of the peaks (or depth of the valleys) from the equilibrium position. Calculate the amplitude given the results of the equilibrium position, above. Note that the decreasing amplitude will render these calculations somewhat inaccurate.

3. The angular frequency is the number of cycles that occur within 2\( \pi \) time and is expressed in radians per second. For these data it’s probably easier to measure the period of oscillation and calculate the angular frequency from the period. The period, \( P \), is the time it takes the function to complete a single cycle. Determine the period from your data and calculate the angular frequency, \( \omega \) from the relation below.

\[
\omega = \frac{2\pi}{P}
\]

4. The phase angle, \( \phi \) is related to the “distance” in time that the sine wave is translated in \( t \) from it’s nominal (un-translated) position. If \( t_1 \) is this time value then the phase angle, \( \phi \), is given by:

\[
\phi = -t_1 \cdot \omega
\]

Note that to shift the curve to the right, the time offset \( t_1 \) must be taken as negative. (If you’re wondering about this, plot \( y = \sin(x - t_1) \), where \( t_1 \) is a constant, and notice what happens to the sine curve as you vary the value of \( t_1 \) from positive to negative.)

5. How do these calculated parameter values compare to those determined by “eyeballing” the fit?
Part 2 – Finding average velocities using a piecewise linear function

6. Make and populate a column in your Excel worksheet to calculate the average velocity between time points. Try remembering the formula for average velocity - check your memory with the earlier exercise where it’s defined. For the time point associated with each average velocity value take the average of the end points of the associated time interval.

7. Plot these velocity data in the same Desmos file as your position data. (There is a folder in the Desmos template for “Velocity Data.”)

8. Display only the following functions or points:
   
   (a) The position function

   (b) The “Segment” function What do you think the “Segment” function is showing? (Try adjusting the parameter “N”.)

9. Why is the “Segment” function called “piecewise linear”?

10. Now also display the velocity data and the “Slope points”. What are the slope points? (Again, try adjusting the parameter “N”.) How do the slope points relate to the velocity data?

11. What type of function do you think will model these data?
12. Enter the function you came up with in item 11, above. Place it as shown below within the “Undamped Velocity” folder in your Desmos file.

\[ y_{v2} = \text{enter function here} \quad \{a \leq x \leq b\} \]

13. You probably need to define different values for some of the parameters. Go ahead and define different constants and give them appropriate values so as to best match your function with your data.

14. How does the phase angle for the velocity function compare to that of the position function? What trigonometric function differs from the sine function by the value that the two phase angles differ? What does this imply about how you can express the velocity function.

15. How does the amplitude of the velocity function differ from that of the position function? What is the value of the ratio of the two?
Part 4 – Modeling the damped oscillation

16. We’ve admittedly been stretching the truth a bit in saying that our data can be truly modeled by the sinusoidal function we’ve been working with. It’s obvious that the noodle’s oscillation is diminishing in amplitude with time. In fact, viewing the video (or just thinking about the situation) tells us that eventually the noodle’s oscillation will die out altogether. How do we modify our model to take this decreasing oscillation into account?

Brainstorm in your group about what type of function decreases in the way that our noodle’s amplitude does. (You might first just think about the peaks of the oscillations. If you had a point at each of the peaks, what function might pass through those successively decreasing peak values?)

17. Plot in Desmos the function you brainstormed above. What parameter(s) do you need in order to best match the function to the peaks of your position data? Adjust these parameters in Desmos to best match your function to the peaks of your data. How might you determine the values of these parameters analytically?

18. Once you’ve matched your new function to the peaks of your position data, copy your sine function and include your new function as a factor of the sine function. That is, your new damped oscillation model will be the product of the original sine function and your new “decreasing function.”

We’re interested in the velocity of the damped oscillating cantilever as well, but we’ll explore finding that function a little later in the course.

\[1\text{Actually the enveloping function doesn’t intersect the damped oscillating function at its peaks. Rather the intersection point is offset from the peaks by } \frac{1}{2} (\sin^{-1}(1) - \tan^{-1}(\frac{1}{k}))\text{, where } k \text{ is the coefficient of } t \text{ in the enveloping function. But thinking about this function as intersecting the peaks will give you a pretty good approximation.}\]
The Product of Velocity and Time - Oscillating Function

Assignment Overview

In a previous worksheet, we found that the instantaneous rate of change of position with time (velocity) for the tip of an oscillating noodle could be expressed as a cosine function (where position vs. time was expressed as a sine function).

As we did with the falling object, we’re now going to turn that discussion around. Suppose we start with an expression for the velocity of the tip of the oscillating noodle. How might we obtain the related position information? That’s what we’ll explore in this exercise.

We’ll begin with the average velocity data we calculated for the oscillating noodle and the velocity function we determined best modeled that data.

That function should have looked something like this:

\[ v(t) = C_1 \cos(\omega t + \phi) \]

Where:

\[ C_1 = \text{amplitude} \quad (1) \]
\[ \omega = \text{frequency} \quad (2) \]
\[ \phi = \text{phase angle} \quad (3) \]

Recall that the position at any time \( t_2 \) is given by the following recursive relation that was derived in a previous exercise on the falling object:

\[ y(t_2) = (t_2 - t_1)v_{\text{average}} + y(t_1) \]

In order to easily plot this in Desmos, we’ll recalculate position, but this time, rather than use our average velocities for each time point that we calculated from the position data, we’ll use velocities calculated from the velocity function. Our plot in Desmos will look something like figure 1, below.

The area of each of the rectangles between the velocity function and the time axis is the product of the elapsed time \( \Delta t \) and the velocity at the point of intersection with the function and is equal to the incremental position change during the elapsed time, \( \Delta y = \Delta t \cdot v_{\text{average}} \). The sum of the areas of all of the rectangles gives us an estimate of the displacement for our falling object. We can write this sum this way for a situation with 20 time intervals:

\[ y_{\text{total}} \approx \sum_{i=1}^{20} (t_{i+1} - t_i)v_{i+1} \]

If we increase the number of rectangles with a concomitant decrease in the elapsed time interval we will more closely approximate the area between the function and the time axis - we’ll eliminate the under and over estimates. As \( \Delta t \) approaches zero (number of rectangles approaches infinity) the limit of this sum of the areas will approach the actual position change for the overall interval of the velocity function.

\[ y_{\text{total}} = \sum_{i=1}^{\infty} (t_{i+1} - t_i)v_{i+1} \]
Learning objectives for this assignment

| 1. Calculate position as the product of average velocity and a time interval. | Part 1 |
| 2. Use the velocity function to calculate position | Part 2 |
| 3. Sum multiple products of average velocity and time intervals to give displacement. | Parts 3 and 4 |
| 4. Understand that the limit of that sum, as the time intervals approach zero and the number of terms becomes infinite, gives the displacement. | Part 3 |
| 5. Visualize the product of average velocity and a time interval as a rectangle between the velocity function and the time axis. | Part 2, step 8 through part 3. |
| 6. Visualize displacement as the sum of multiple rectangles. | Part 4 |

Steps to complete the assignment

Part 1 – The product of velocity and time

1. Open your Excel worksheet from the previous oscillating noodle exercise.

2. In a column just to the right of the column of position data you were using previously, calculate the incremental distance traveled for each time interval using the position data. Label the column appropriately. (Insert a column if necessary.)

3. In a column just to the right of the column you added above, calculate the incremental distance traveled for each time interval, but this time using the velocity values.

   Note that the two columns should have identical values. Why do you think this should be?

4. In a column to the right of the other two, calculate the cumulative position of the falling object. These values should be identical to your original position data.

5. In your words describe:
   (a) What this cumulative position data represents, and
   (b) How it was calculated in the step above.

Part 2 – Using the velocity function to calculate position.

We’re going to recreate the cumulative position data but instead of using the velocities you calculated from the position data (as you did in part 1) you’ll use velocities calculated from the velocity function.

6. In a new column calculate the cumulative position of your falling object using velocities based on the velocity function. Use the velocity function from your Desmos oscillating noodle file.

7. Plot this new position data in a graph along with the velocity data and the velocity function. (You can copy the existing graph and modify it, if you like.)

8. In your Desmos file for the oscillating noodle input, ensure that the following parameters are set as specified below.
   (a) Set slider a to the initial time value in your table and slider b to the last time value (in the “Segments” folder).
   (b) Set slider n (the number of time increments or rectangles) to match the number of increments in your table (in the “Formulas” folder).
   (c) Set slider c to 0.5. This sets the rectangles so that the top center sits on the velocity function (in the “Segments” folder).

10. What do the shaded rectangles represent? It may be helpful to refer back to figure 1 above.

Part 3 – Increasing the number of time increments

11. How does the sum of the incremental distances found in Desmos compare to the displacement in your initial data? (See the “Summation” folder).

12. In Desmos increase (or decrease) the value of \( n \) and notice what happens to the change in the total area (displacement).

13. Do you think this will always be the case? Explain.

Part 4 – Summing the areas of the rectangles to find position

14. Make sure that the values of “\( N \)” and “\( n \)” in your Desmos file are set to the number of time intervals in your recorded data. Calculate the change of \( y \) position for the first time interval based on the left-most rectangle in your Desmos file.

15. Repeat this calculation for the next time interval. What is the cumulative \( y \) position change of the object?

16. In your Desmos file display the items in folder “Position Points.” Interrogate the left two points for their coordinates. What does this new set of points represent?

17. Display the position function. Discuss the relationship between the position function and the newly displayed set of points. Do your “position points” follow the position function?
18. Your “position points” may follow the contour of your position function, but not lie on the function curve. Why might this be? Where is the position curve at \( t = 0 \)? Where is the “position point” for \( t = 0 \)? How might you correct the position of your “position points”?

19. Describe the process involved in each of the following steps you went through in the “oscillating noodle” exercises:
   
   (a) Recording the \( y \) position vs. time data of the tip of the cantilevered noodle.

   (b) Finding a function that models the noodle tip’s instantaneous position.

   (c) Finding the average velocity of for each time interval of recorded position data.

   (d) Finding a function that models the object’s instantaneous velocity.

   (e) Summing the “areas” between the velocity function and the time axis to arrive back at values for the object’s position.

20. How might you determine a function that models the object’s position if you were starting with the velocity function?
Part 5 – Presenting the results of your analysis

21. Organize your answers to items 20 and 21 above into a 5 minute presentation:
   (a) What key points do you want to make (select 3 or 4)?
   (b) How will you best communicate these points?

22. Select a spokesperson who will present your results when selected to do so.

Post-assignment reflection

1. Review the Learning objectives for this assignment toward the beginning of this document. Write down the Learning Objectives that you fully understand.

2. Write down the Learning Objectives that you are unsure about or have questions about. Seek out help to answer your questions or further your understanding from one or more of these resources:
   (a) One of your fellow group members or other colleague. (You might use the class Discussion Board for this.)
   (b) The web.
   (c) The Center for Academic Excellence.
   (d) Your instructor (either by email or at office hours).