Notes on Parabolas Using the Mirage Illusion

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Introduction
The present work is intended as a classroom note on the topic of parabolas. We present several real-world applications of parabolas, outline a short classroom lab activity using the Mirage Illusion, derive fundamental formulas and properties of parabolas, and suggest analogous discussions in the context of ellipses. The content is suitable for use in college algebra or precalculus as well as in first-year calculus courses.

Motivation and Applications of the Parabola
An important geometric characteristic of parabolas is that entering rays are reflected on the sides and redirected toward a common point—the focus. Receiving devices such as telescopes, some solar power plants [4], listening devices such as the whispering parabolic dishes at the California Science Center in Los Angeles [5], and satellite dishes utilize this property to concentrate light, sound, and other signals at their focus. The inverse property—rays originating from the focus form a beam exiting the parabola—are used in transmission devices such as flashlights, headlights, and parabolic heaters. Parabolic heaters use a computer-designed parabolic reflector to focus heat, just like a satellite dish concentrates TV signals, to warm in the direction it is pointed. Interestingly, the word focus originates from the Latin word for domestic hearth or fireplace [3].

Classroom Activity with the Mirage Illusion
The Mirage Illusion by Opti-Gone International [2] used in this article employs two paired paraboloidal mirrors to create a three-dimensional hologram of an object placed inside the apparatus. The object is placed at the bottom of the lower mirror, which by design coincides with the focus of the upper mirror. The image is displayed through an opening at the vertex of the upper mirror, which coincides with the focus of the lower mirror (see Figure 1). The Mirage currently sells in the $40 range. Numerous devices like Mirage can be found on the Internet. Perhaps one can even be made with rapid prototyping (3-D printing).

After a presentation on parabolas that includes their equations and a discussion of the geometric properties of the focus, students can be shown the Mirage Illusion and asked to discover how it works in small groups. Students are amazed at the realistic looking 3-D real image. Necessary materials for this discovery activity include a ruler and a Mirage Illusion device for each group. Obviously, this is a fairly open-ended question and, hence, some suggestions of how to begin the investigation process might be required from the teacher.

A central goal of the activity is for students to write an equation of the vertical cross section of the mirror passing through the vertex (see Figure 2). From that equation, they can calculate the location of the focus and verify that the focus of each mirror coincides with the vertex of the other.

A derivation of an equation of Mirage Illusion cross section might proceed as follows. Recall that a parabola with the $y$-axis as its axis of symmetry, having its focus at $(0, p)$, and the line $y = -p$ as its directrix (hence, having its vertex at the origin) can be expressed by the following equation (see Figure 3).

\[ x^2 = 4py \]
This formula can be derived from the definition that a parabola consists of all points \((x, y)\) that are equidistant from a fixed point \((0, p)\) (the focus) and a fixed line \(y = -p\) (the directrix).

Expressing and equating these distances yields the equation

\[
y + p = \sqrt{x^2 + (y - p)^2}
\]

Squaring each side yields \((y + p)^2 = x^2 + (y - p)^2\) or \(y^2 + 2yp + p^2 = x^2 + y^2 - 2yp + p^2\), which simplifies to equation (1). This may be a worthwhile algebra exercise.

Each paraboloidal mirror has a diameter of 9 inches and a depth of 1.5 inches [2]. The students approximate these dimensions by actually measuring the Mirage with a ruler.

If we represent the cross-section as a parabola in the \(xy\)-plane, with vertex at the origin, and opening upward, then this data can be interpreted by saying that the parabola passes through the point \((4.5, 1.5)\). Substituting this point into equation (1) allows us to find the value of \(p\).

\[
x^2 = 4py
\]

\[
(4.5)^2 = 4p(1.5)
\]

\[
20.25 = 5p
\]

\[
p = 3.375
\]

This means that the focus of the lower mirror is 3.375 inches above its vertex, which is consistent with the distance between the object and its image.
In this section, we will use the physical fact that the angle of incidence is equal to the angle of reflection to prove that a ray entering a parabola parallel to its axis of symmetry will be reflected toward its focus. This is the principle fact explaining why/how the Mirage Illusion works.

In Figure 6, line \( I \) represents an incoming ray parallel to the \( y \)-axis, \( Q \) is the point of contact between that ray and the parabola given by \( x^2 = 4py \), line \( T \) is the line tangent to the parabola passing through \( Q \), and point \((0, p)\) is the focus of the parabola. Line \( R \) represents the path of the reflected ray and it is defined by the fact that it contains \( Q \) and is oriented such that \( m\angle DQE = m\angle BQC \) (since the angle of incidence equals the angle of reflection).

Figure 4. Measurement of the diameter at the top of the paraboloid

Figure 5. Measurement of the depth of the paraboloid using a second ruler as a guide

Parabolic Reflection Principle

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Proposition 1. Referring to Figure 6, line $R$, which is defined as the line through point $Q$ with slope such that $m\angle DQE = m\angle BQC$, passes through point $(0, p)$.

Proof. Our approach to proving this will be to construct the equation of line $R$ and show that it is satisfied by the point $(0, p)$. Begin by noting that since line $T$ is tangent to the graph of $x^2 = 4py$ at $Q$, we can compute its slope by using the derivative. From this, we will find the measure of $\angle BQC$.

This is a good application of trigonometry. Therefore, an equation for line $R$ in point-slope form is

$$y - x_0^2 = \frac{x_0 - p}{4p} (x - x_0),$$

which is easily seen to be satisfied by the point $(0, p)$.

Remark. If this proof were to be presented in a precalculus class, then the computation using the derivative in the above proof could be replaced by a direct computation of slope using the difference quotient.

**Analogy with Ellipses**

Ideas entirely analogous to those presented for the parabola can be presented in the context of ellipses. An ellipse is characterized by the geometric fact that a ray originating at one of its foci will be reflected by the ellipse toward the other focus. A proof of this fact is analogous to the proof of the preceding proposition.

Although, the authors are unaware of a convenient classroom device analogous to the Mirage Illusion that illustrates or utilizes this property of ellipses, there are certainly many examples of real-life ellipse applications. One such example is the medical device called the lithotripter. A lithotripter is a half ellipsoid-shaped device that pulverizes kidney stones. The patient’s kidney stone must be at one focus point and the shockwave generator at the other focus. The lithotripter, if successful, can break up large kidney stones into smaller ones that can be passed by the patient, thus avoiding dangerous surgery. Another fun application is an elliptical pool table; one of the authors built one and used it in classroom demonstrations for decades. In an elliptical pool table, if a black spot were at $F_1$ and a hole (pocket) at $F_2$, shooting a ball (with no spin) through $F_1$ would always make the ball ultimately land in the pocket, $F_2$ (see Figure 6.).
Figure 7). A classroom activity in the context of ellipses could be to ask students to draw their own ellipses using pegs (for the foci) and a string [1]. Justification for such a construction follows from the popular definition that an ellipse is the collection of all points in the $xy$-plane whose sum of distances to two fixed points (foci) is constant ($PF_1 + PF_2 = 2a$). It is a good review of algebra to derive the equation of an ellipse from this definition. The reflective property of ellipses will be visible during the construction by noting that $m\angle APF_2 = m\angle BPF_1$ (see Figure 7).

Figure 7.

Statuary Hall in the U.S. Capitol building is elliptic. It was in this room that John Quincy Adams, while he was still a member of the House of Representatives, discovered the following acoustical phenomenon: when he situated his desk at a focal point of the elliptical ceiling, he could easily eavesdrop on the private conversations of other House members located near the other focal point. This is a popular demonstration during the Capitol Tour.

References


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For the past 41 years, Sid Kolpas has taught at the junior high school, high school, and college levels. For the past 20 years, he has been a professor of mathematics at Glendale CC in Glendale, CA. In 2010, he received the Hayward Award for Excellence in Education from the California Community Colleges Board of Governors. In fall 2011, he started a new career at Delaware County CC, outside of Philadelphia.