Making Pi*e* from Scratch
Rapidly and Memorably

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Abstract

This webinar will examine several iterative methods for generating fractions to represent pi better and better, culminating in an iterating brackets method. In addition, the speakers will introduce various theorems relating to reliability and convergence, identify some easily remembered approximations, extend the iterating brackets method to all irrationals [6], and consider the approximation's behavior with rational targets.

Motivating

You likely share our lifelong fascination with $\pi \stackrel{\text{def}}{=} \frac{C}{d}$, the ratio of a circle's circumference to its diameter. Our first teachers introduced us to $\frac{22}{7}$ and 3.14 as estimates for π. A college lecturer offered a significant improvement with a clever mnemonic:

Write the repeated odd digits $113\backslash355$ with a backslash between the 3*s,* then read the fraction as $\frac{355}{113}$. This memorable "Chinese estimate" gives seven digits of π; Archimedes' $\frac{22}{7}$ only three correct digits, no better than 3.14. Could we find even better approximations, yet quick and easy to recall?

π

The fraction 165,707,065/52,746,197 = 3.141592653589793 4 (16 correct digits) is one of our own striking π-approximations. Naturally, we notice repeated 65*s* and 70*s,* predictably craving connections and spotting patterns, even when none exist. We break this fraction into mnemonic pieces—(1,65,70,70,65)/(5+2=7,4+6-1=9,7)—but must memorize an extra digit!

In fact, McLoone calculates only about a one-to-one return on mental investment, warning in his blog, *All rational approximations of pi are useless.* [7] (That didn't deter us.)

Elaborating, in their 2002 article *Surprisingly accurate rational approximations* [2], Apostol and Mnatsakanian establish that every irrational has a close rational approximation, with about half the number of digits in its denominator as the number initially correct in its decimal approximation. Unfortunately, that gives us no fewer digits to memorize.

In their 2001 article *Good rational approximations to logarithms* [1], Apostol/Mnaksakian use weighted mediants to approximate $log_{10} k$. Their notation and strategy are similar to those that we adopt independently. Wisner/Herzinger [5, 9, 10] and Gilberson/Osler [4] discuss related ideas, like continued fractions or Greek ladders (correlations in the Appendix).

Our experiences can inform you

Although our own journey begins simply enough, it advances through increasingly complex patterns and more diverse fields like vector addition, weighted mediants, mathematical induction, and explementary angles.

You too may bravely look to try your hand at a research paper. Here we will highlight several primary principles learned through the paper-writing process. We will pull aside the curtain, revealing behind-the-scenes stuff:

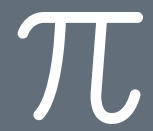
Principles that we learned

- Your paper may end up looking “pretty,” but it never starts out that way.
- You will experience frustration, facing virtual roadblocks for extended times.
- Expect these to be followed by the amazing high of breakthroughs.
- Such patterns can repeat multiple times as you grow the paper.
- Working with a colleague helps you through hard times, gives you perspective.
- You can learn to view the developing results through the eyes of the reader.
- A collaborative paper is an ideal way to energize your most insightful students.

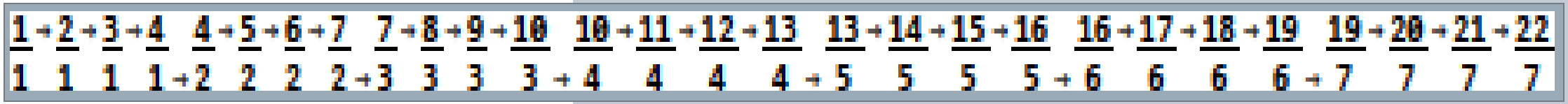
Incrementing numerator or denominator – 1st method

An irrational such as π cannot *exactly* equal a rational $\frac{n}{d}$. Increasing the numerator or denominator of a fraction results in a larger or smaller fraction, respectively. These ideas suggest a simple iterative algorithm to approach π, shown in Figure 2:

If $\frac{n}{d} < \pi$, increment $n^{\rightarrow}$, else increment $d_{\rightarrow}$



First fractions generated by algorithm in box, visualized by high and low arrows, leading to 22/7.



Restated: If $\frac{n}{d} < \pi$, increment $n^{\rightarrow}$, else increment $d_{\rightarrow}$

Figure 2.

Severely limited

This algorithm becomes the one-line *Mathematica* code:

For $[n = d = 1, n < 100(* \, or \; any \; limit \, *), Print[n, "/", d]; If[{}^{n}/_{d} < Pi, n++, d++]]$

In Table 1 (five left-hand columns), consider the large number of steps necessary to reach the next fraction of merit $\frac{n}{d}$. The sequence converges gradually and reaches the familiar $\frac{22}{7}$ at step 28; then accuracy degrades as usual.

In Figure 3, notice how sporadic this approach is. We absolutely need a better method with more speed and less wasted effort.

1st method: Increment n or d,

but only steps to increment d.

To reach fraction n/d:

Start at $1/1$, then step by step, increment numerator or denominator

Trying every n and d is slow!

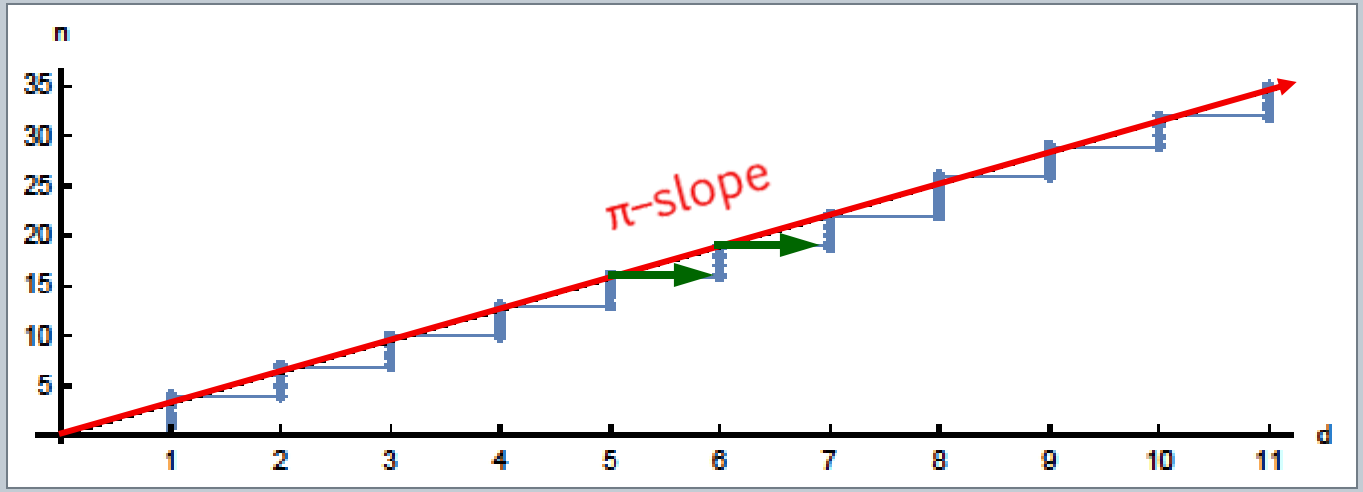
Table 1.

Focus on purple information

Step	n/d	decimal	digits	error	
03	3/1	3.00000	1	-0.14159	
07	6/2	3.00000	1	-0.14159	
11	9/3	3.00000	1	-0.14159	
16	13/4	3.25000	1	+0.10841	
20	16/5	3.20000	1	+0.05841	
24	19/6	3.16667	2	+0.02507	
28	22/7	3.14286	3	+0.00126	
…	…	…	…	…	
115	88/28	3.14286	3	+0.00126	

Increment **numerator** n or **denominator** d, by plot

Trying every n and d is sporadic and saw-toothed!



As n **is incremented vertically**, the graph approaches the **π-slope** from below until it over-shoots that target. But it slides right, far below π each time d **is incremented horizontally**.

Figure 3.

π

Incrementing denominator only – 2nd method

Fortunately, we can readily find the optimal numerator choice for each denominator. Together, these form the best rational π-approximation (for that given denominator), now called a ***π-approximant***:

Multiply by π each positive-integer denominator d. Round this product to get the nearest integer numerator n (*Mathematica* code: ***n***=Round[***d*** Pi]). Table 2 (right-hand steps) lists the elements in this sequence of all π-approximants.

Increment d – 2nd method

List only steps to increment d.

To reach fraction n/d

Left: Increment numerator or denominator

Right: Increment denominator

For $22/7$, this is $28 \div 7$ = four times faster.

Table 2

Left: starting at $1/1$, increment numerator or denominator for step $(n + d - 1)$

Focus on yellow information

Step	n/d	decimal	digits	error	Step
03	3/1	3.00000	1	-0.14159	01
07	6/2	3.00000	1	-0.14159	02
11	9/3	3.00000	1	-0.14159	03
16	13/4	3.25000	1	+0.10841	04
20	16/5	3.20000	1	+0.05841	05
24	19/6	3.16667	2	+0.02507	06
28	22/7	3.14286	3	+0.00126	07
…	…	…	…	…	…
115	88/28	3.14286	3	+0.00126	28

Right: starting at $3/1$, increment only the denominator for each step d.

Errors and relative errors

The second method is about four times faster than the first, but convergence is still frustratingly slow. Accuracy gains are nonuniform, unsatisfactory. But errors are limited:

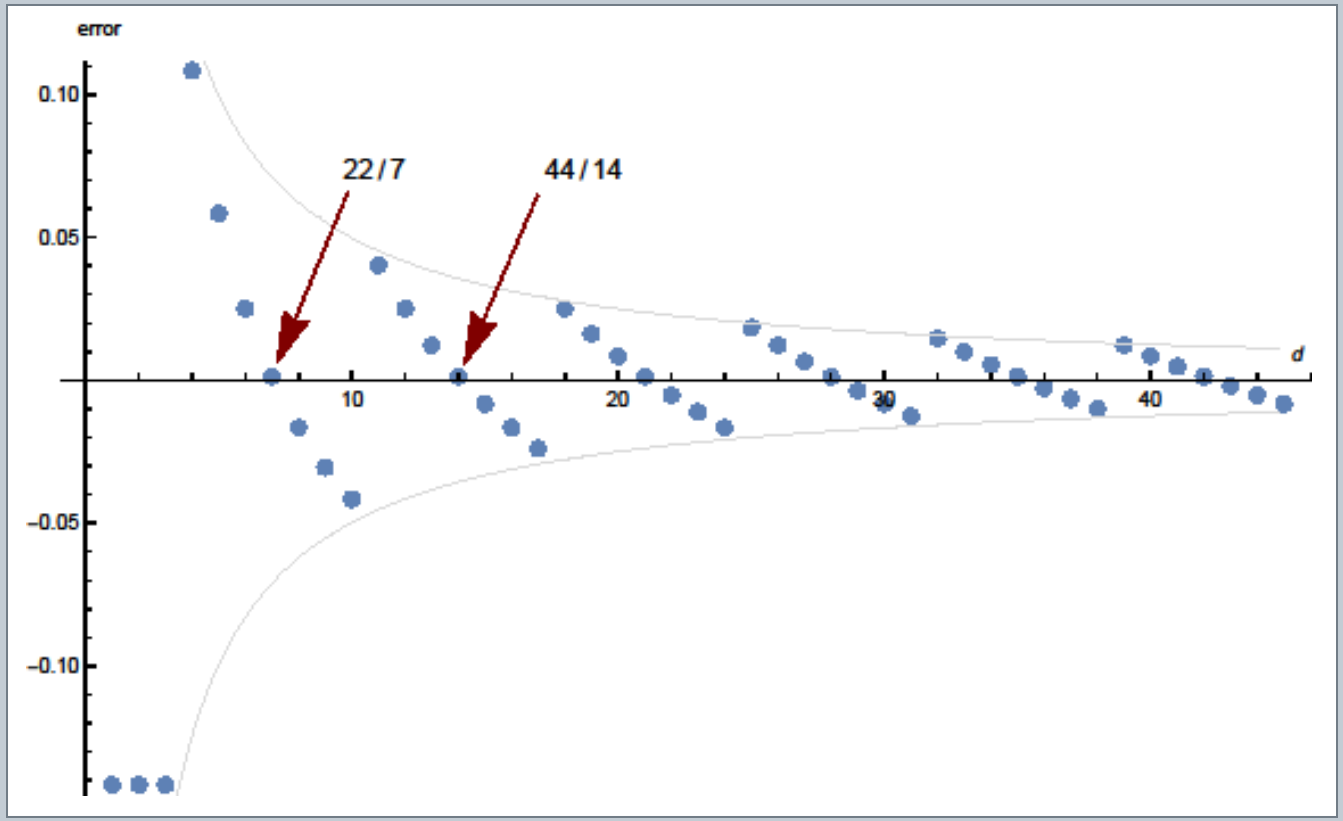
- $Error \stackrel{\text{def}}{=} \frac{n}{d} - \pi$, where $n \stackrel{\text{def}}{=} Round[d\,\pi]$, which means
- $-0.5 < (n - d\,\pi) < 0.5$, implying two error bounds:

1) $|Error| \stackrel{\text{def}}{=} \left|\frac{n}{d} - \pi\right| < \frac{0.5}{d}$

2) $Relative\ error \stackrel{\text{def}}{=} \frac{error}{max|error|} = \frac{\frac{n}{d} - \pi}{\frac{0.5}{d}} = \frac{\frac{Round[d\,\pi]}{d} - \pi}{\frac{0.5}{d}}$ or $2 \cdot (Round[d\,\pi] - d\,\pi)$

Figure 4 shows *errors* for the first forty-five denominators. Figure 5 illustrates *relative errors* for one hundred twenty denominators.

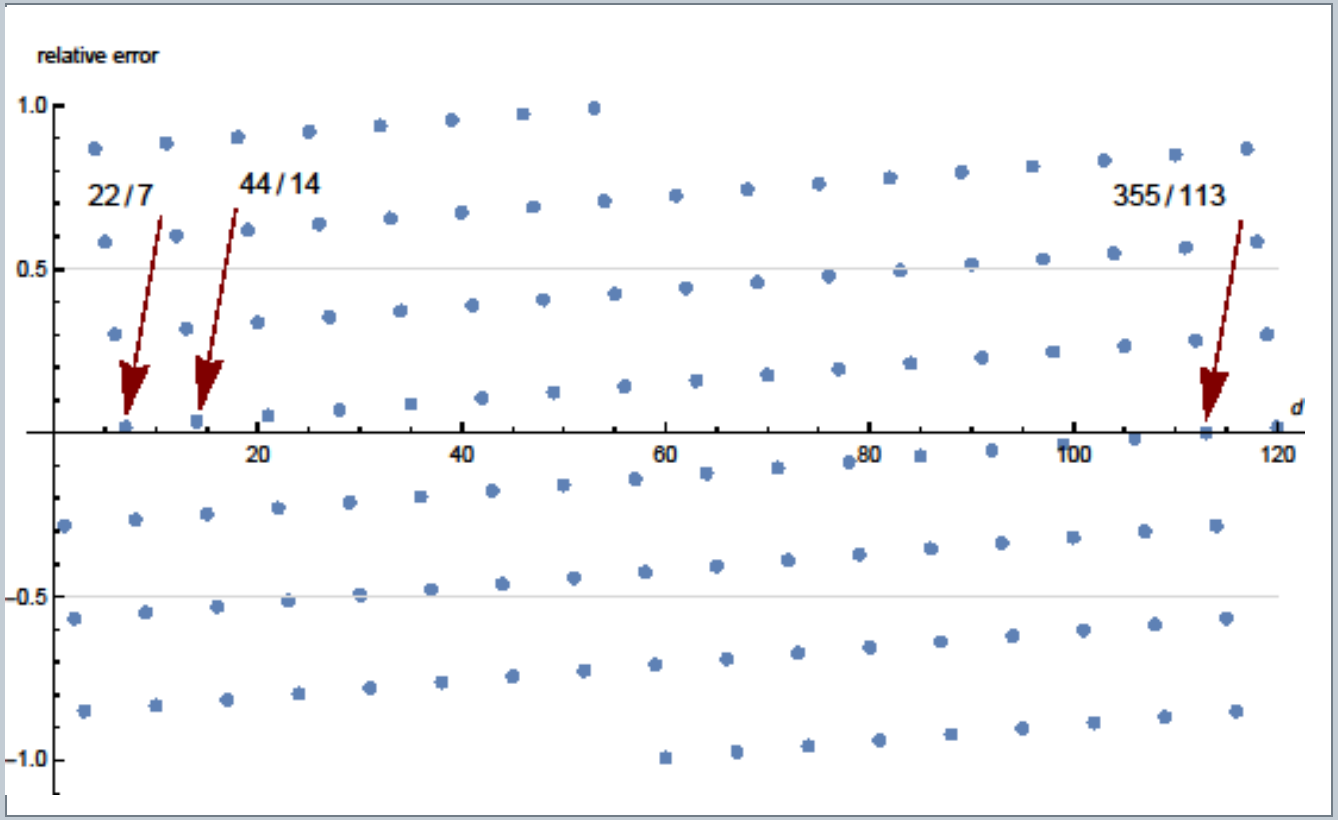
With no distinction between $22/7$ and $44/14$:



Errors for the first forty-five integer denominators, showing the $\frac{\pm 0.5}{d}$ bounding envelope. Points closer to the horizontal axis, like $22/7$, $44/14$, $66/21$, are better fractional approximants.

Figure 4.

With clear difference between $22/7$ and $44/14$:



Relative errors for one hundred twenty integer denominators: Notice that $44/14$ is visibly worse than $22/7$. (Its error is a larger fraction of the maximum absolute error, $\frac{0.5}{d}$.) Far superior is $355/113$.

Figure 5.

Saving improvements – 3rd method

Next, we enhance the algorithm *n*=Round[*d* Pi] of Table 2 (2nd method) by adding only *improved* elements to the list of fractions that we track. But an increasingly complex *Mathematica* algorithm is needed to identify these fractions and to create the *gold* (standard) *list* of all better π-approximants, shown in Table 3:

```
BestApproximations={3/1};
For[d=1,d<= 10^5(*or any limit*),d++,
   n=Round[d Pi];
   If[Abs[n/d - Pi]<Abs[Last[BestApproximations]-Pi],
      AppendTo[BestApproximations,n/d]]];
BestApproximations
```

Gold list – 3rd method

- Column n/d contains recent "good" approximants
- Number of correct digits slowly increases
- |Error| slowly decreases
- Wow: Best n/d entries migrate to & repeat in $\Delta n/\Delta d$ column!!

Table 3.

step	n/d	digits	+/- error	$\Delta n/\Delta d$	improv.
01	3/1	1	-1.42E-1	--/--	---
02	13/4	1	1.08E-1	10/3	1.3
03	16/5	1	5.84E-2	3/1	1.9
04	19/6	2	2.51E-2	3/1	2.3
05	22/7	3	1.26E-3	3/1	19.8
06	179/57	3	-1.24E-3	157/50	1.0
07	201/64	3	-9.68E-4	22/7	1.3
08	223/71	3	-7.48E-4	22/7	1.3
09	245/78	3	-5.67E-4	22/7	1.3
10	267/85	3	-4.16E-4	22/7	1.4
11	289/92	4	-2.88E-4	22/7	1.4
12	311/99	4	-1.79E-4	22/7	1.6
13	333/106	4	-8.32E-5	22/7	2.1
14	355/113	7	2.67E-7	22/7	312.0
15	52,163/16,604	7	-2.66E-7	51,808/16,491	1.0
16	52,518/16,717	7	-2.63E-7	355/113	1.0
17	52,873/16,830	7	-2.59E-7	355/113	1.0

An unexplained oddity leads to a real breakthrough by asking the right questions:

- *Why* do highlighted fractions in the n/d column reappear in the $\Delta n/\Delta d$ column?
- *How* does "adding" $^{3}/_{1}$ three times to $^{13}/_{4}$ result in $^{22}/_{7}$?

$$^{13}/_{4} + \underbrace{^{3}/_{1} + {}^{3}/_{1} + {}^{3}/_{1}}_{3\ \Delta\text{-terms}} = {}^{22}/_{7}$$

- *When* will "adding" $^{22}/_{7}$ eight times to $^{179}/_{57}$ give $^{355}/_{113}$?

$$^{179}/_{57} + \underbrace{^{22}/_{7} + \cdots + {}^{22}/_{7}}_{8\ \Delta\text{-terms}} = {}^{355}/_{113}$$

- *What* pattern predicts how many times to "add" $^{355}/_{113}$ resulting in what fraction?

Table 4 (Table 3, reconsidered)

step	n/d	digits	+/- error	$\Delta n/\Delta d$	improv.
01	3/1	1	-1.42E-1	--/--	---
02	13/4	1	1.08E-1	10/3	1.3
03	16/5	1	5.84E-2	3/1	1.9
04	19/6	2	2.51E-2	3/1	2.3
05	22/7	3	1.26E-3	3/1	19.8
06	179/57	3	-1.24E-3	157/50	1.0
07	201/64	3	-9.68E-4	22/7	1.3
08	223/71	3	-7.48E-4	22/7	1.3
09	245/78	3	-5.67E-4	22/7	1.3
10	267/85	3	-4.16E-4	22/7	1.4
11	289/92	4	-2.88E-4	22/7	1.4
12	311/99	4	-1.79E-4	22/7	1.6
13	333/106	4	-8.32E-5	22/7	2.1
14	355/113	7	2.67E-7	22/7	312.0
15	52,163/16,604	7	-2.66E-7	51,808/16,491	1.0
16	52,518/16,717	7	-2.63E-7	355/113	1.0
17	52,873/16,830	7	-2.59E-7	355/113	1.0

Noticing step differences

This anomaly disturbingly resembles fraction “addition” by adding numerators and adding denominators, a common error of struggling, basic students. Yet these questions lead to an extended, accelerated method for generating all π-approximants!

Forming mediants

The operation of adding numerators and denominators separately is called, officially, the mediant; disparagingly, freshman sum; and alternatively, Farey mean [9].

Definition 1. *The **mediant** of the **progenitor** fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$ *is defined:*

$$\frac{a}{b} \oplus \frac{c}{d} \stackrel{\text{def}}{=} \frac{a+c}{b+d}$$

Property 1 (Mediant inequality). *If* $\frac{a}{b} < \frac{c}{d}$, *where* $b, d > 0$, *then*

$$\frac{a}{b} < \frac{a}{b} \oplus \frac{c}{d} < \frac{c}{d}.$$

So the mediant lies strictly between its progenitors.

For example: $\frac{1}{3} < \frac{2}{5} < \frac{1}{2}$; that is, $0.333\cdots < 0.4 < 0.5$.

Derivation. See Supplementary Section.

π

When do approximants pass on their name property to their mediant "child"?

Theorem 1 (Mediant of approximants). *When π-approximants* $\frac{a}{b}$ *and* $\frac{c}{d}$ *are on opposite sides of* π*, their mediant is likewise a π-approximant.*

Proof. See Supplementary Section.

In general, it can be shown that one of these three fractions with consecutive numerators is a π-approximant for the denominator $(b + d)$:

$$\frac{(a+c)-1}{b+d}, \frac{a+c}{b+d}, \frac{(a+c)+1}{b+d}.$$

The mediant lies strictly between its progenitors.

Mediant in vector form:

- $\vec{v} \stackrel{\text{def}}{=} (b, a)$ and $\vec{w} \stackrel{\text{def}}{=} (d, c)$
- $\overrightarrow{v + w} \stackrel{\text{def}}{=} (b + d, a + c)$

Notice the π-slope line:

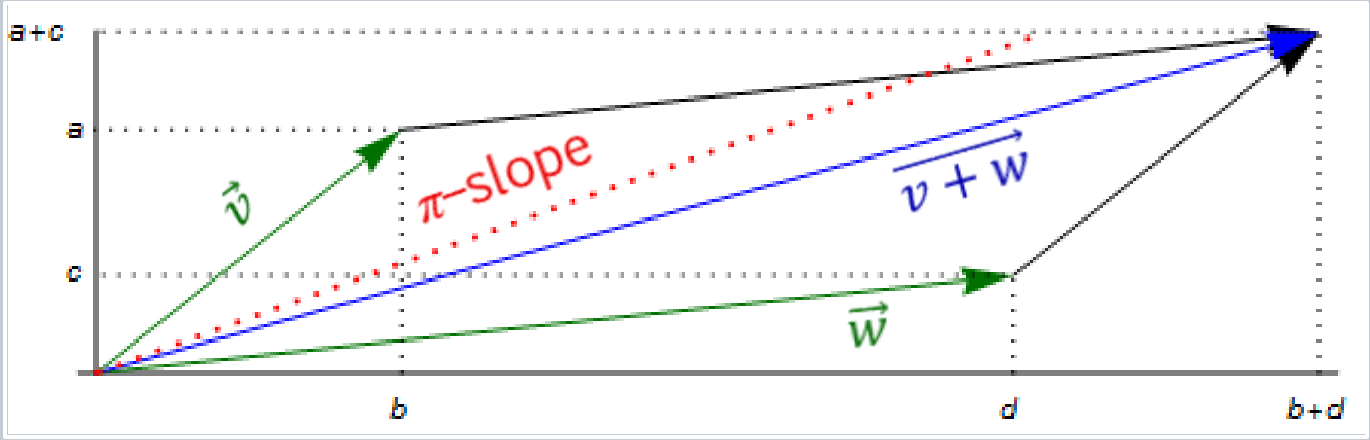
- $y = \pi x$, not to scale

Mediant in slope form:

- $v + w \stackrel{\text{def}}{=} \frac{a}{b} \oplus \frac{c}{d} \stackrel{\text{def}}{=} \frac{a+c}{b+d}$.

Figure 6.

GEOMETRY ILLUSTRATES ALGEBRA



The vector $\overrightarrow{mediant\ v + w}$ lies between $\vec{v}$ and $\vec{w}$, its vector $\overrightarrow{progenitors}$. Assume that the $\overrightarrow{progenitors}$ are on opposite sides of the π-slope line. Their $\overrightarrow{mediant}$ then will be closer to π than the $\overrightarrow{progenitor}$ on the same side, here $w = \frac{c}{d}$.

Bracketing pi – 4th method

Definition 2. *The nonnegative fraction pair* $\left\{\frac{a}{b}, \frac{c}{d}\right\}$ *forms a* ***bracket*** *for the positive real-number* ***target*** r *if and only if* $\frac{a}{b} < r < \frac{c}{d}$ *or* $\frac{a}{b} > r > \frac{c}{d}$. *When the denominators are ordered* $0 < b < d$, *the* ***earlier*** *fraction (in the list) is* $\frac{a}{b}$, *conventionally listed first, followed by the* ***later*** *fraction* $\frac{c}{d}$. *Each bracket* $\left\{\frac{a}{b}, \frac{c}{d}\right\}$ *is associated with the interval* $\left(\frac{a}{b}, \frac{c}{d}\right)$ *or* $\left(\frac{c}{d}, \frac{a}{b}\right)$, *here the smaller fraction listed first. We say that the real number* r *is* ***within*** *the bracket* $\left\{\frac{a}{b}, \frac{c}{d}\right\}$ *if it lies inside the appropriate interval. The* ***length*** *of bracket* $\left\{\frac{a}{b}, \frac{c}{d}\right\}$ *is distance* $\left|\frac{a}{b} - \frac{c}{d}\right|$.

For irrational targets, all intervals are open. In the Appendix, we consider the case in which the rational target equals the later fraction.

Now we can reconstruct our gold list simply by repeatedly constructing mediants, which motivates the next step:

Bracketing π – 4th method (see the highlighted entries):

From right-hand steps 1 and 2, "add" 3/1 to 13/4 3 times, to reach 22/7 (via 16/5, 19/6).

Now we only need 3 steps!

Table 5.

Focus on yellow information

Step	n/d	decimal	digits	error	Step
03	3/1	3.00000	1	-0.14159	01
07	--	3.00000	1	-0.14159	--
11	--	3.00000	1	-0.14159	--
16	13/4	3.25000	1	+0.10841	02
20	(16/5)	3.20000	1	+0.05841	--
24	(19/6)	3.16667	2	+0.02507	--
28	22/7	3.14286	3	+0.00126	03
115	88/28	3.14286	3	+0.00126	28

First, we generalize the simple mediant concept.

Definition 3. *The **weighted mediant** of fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$ *is*

$$w \odot \frac{a}{b} \oplus \frac{c}{d} \stackrel{\text{def}}{=} \frac{w \times a}{w \times b} \oplus \frac{c}{d} \stackrel{\text{def}}{=} \frac{wa + c}{wb + d},$$

*where the **weighting factor** is any nonnegative real number* w. *For any nonnegative integer* i, *we call the* i*-weighted mediant simply the* $\boldsymbol{i^{th}}$***-mediant.***

Remark. *Only weight the earlier fraction* $\frac{a}{b}$, *having the smaller denominator. For the full version, see Supplementary Section.*

Second, we extend the mediant inequality concept.

Property 2 (Weighted mediant inequality). *Assume that* $0 \leq v < w$, *where* $a, b, c, d \in \mathbb{N}$ *(positive integers). Then exactly one statement holds true:*

$$\begin{aligned} &\textit{either} \quad \frac{a}{b} < w \odot \frac{a}{b} \oplus \frac{c}{d} < v \odot \frac{a}{b} \oplus \frac{c}{d} \leq \frac{c}{d} \\ &\textit{or} \quad \frac{a}{b} > w \odot \frac{a}{b} \oplus \frac{c}{d} > v \odot \frac{a}{b} \oplus \frac{c}{d} \geq \frac{c}{d} \end{aligned} \tag{1}$$

Derivation. See Supplementary Section.

Remark. *Weighted mediants lie between progenitors. As the weighting index increases, mediants move from the later fraction* $\frac{c}{d}$ *toward the earlier fraction* $\frac{a}{b}$.

From initial bracket $\left\{\frac{3}{1},\frac{13}{4}\right\}$, the weighted-mediant sequence is $\frac{13}{4},\frac{16}{5},\frac{19}{6},\frac{22}{7},\cdots$. These elements belong in the gold list, because each is on the same side of π as $\frac{13}{4}$ and closer to π than its predecessor. Thus, they form a sequence monotonically approaching π from the $\frac{13}{4}$- side (to be proven in Theorem 2).

The situation changes when $4\odot\frac{3}{1}\oplus\frac{13}{4}\left(=\frac{25}{8}\right)$ appears on the other side of π, with the sequence elements now receding from π. However, we have already generated a tighter bracket.

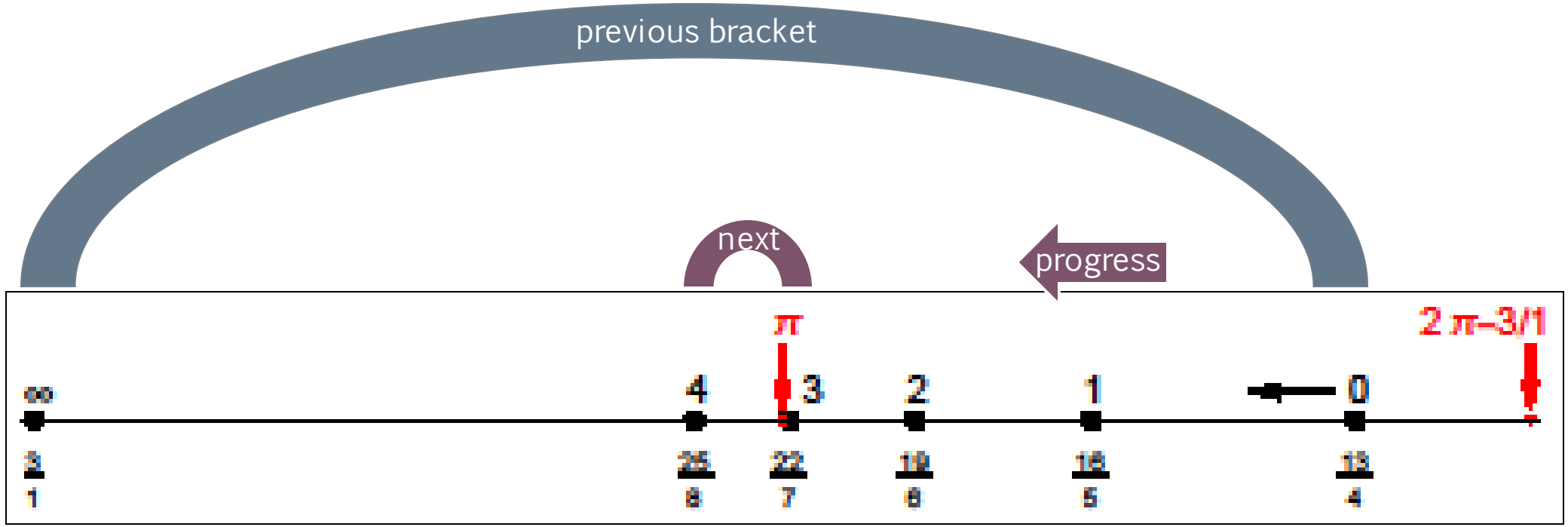


Figure 7.

Weighted mediants advance from $\frac{13}{4}$ toward $\frac{3}{1}$, right to left, reaching the next bracket $\left\{\frac{22}{7}, \frac{25}{8}\right\}$.

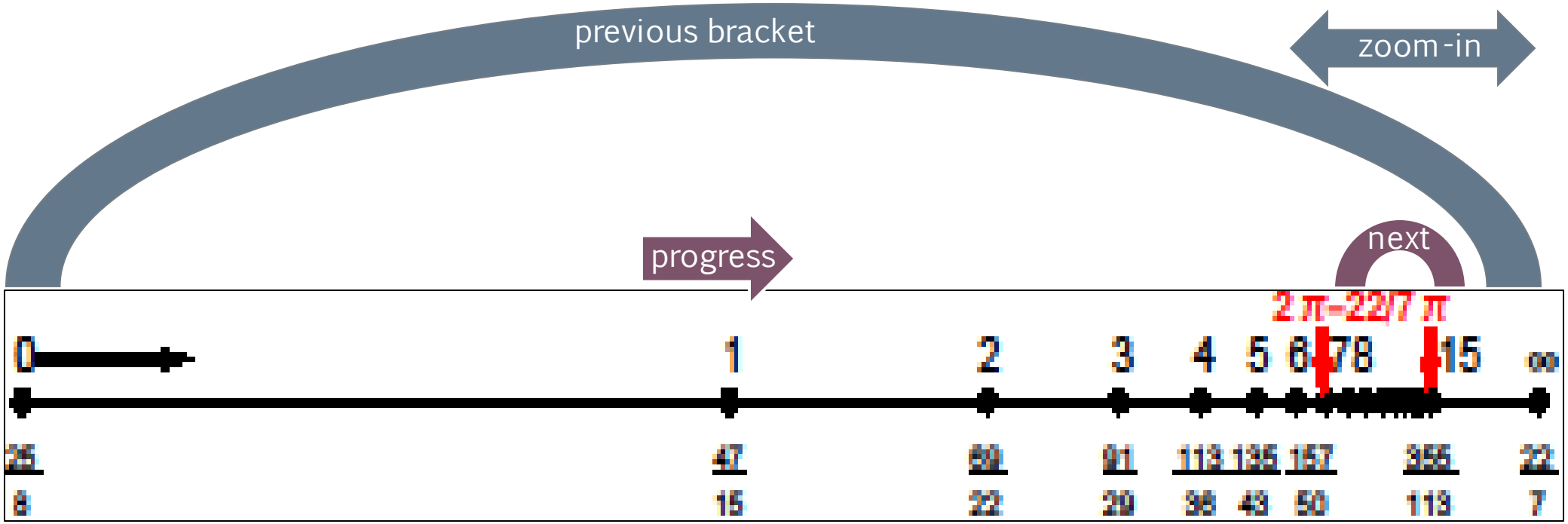


Figure 8.

Mediants progress from $\frac{25}{8}$ toward $\frac{22}{7}$, now left to right, reaching the next bracket $\left\{\frac{333}{106}, \frac{355}{113}\right\}$.

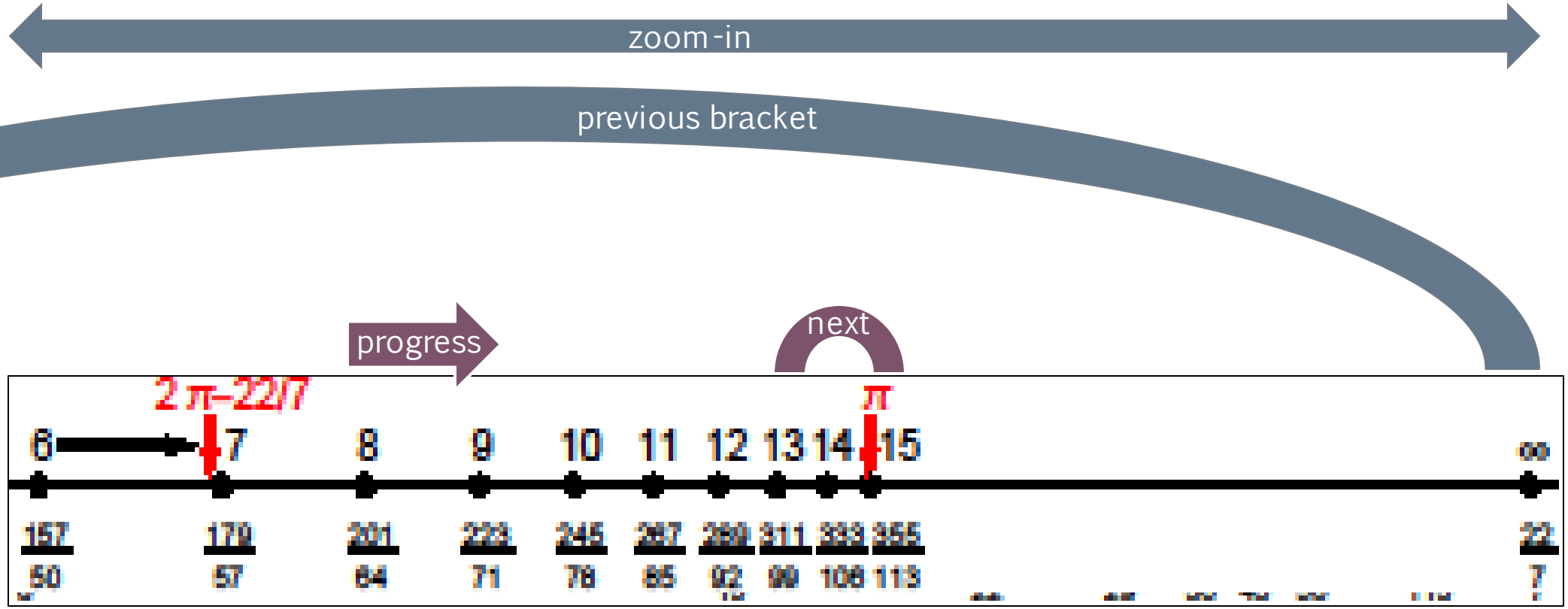


Figure 9.

Zoom: 6th to 15th–mediants, closer to π than previous ones; create next bracket, $\left\{\frac{333}{106}, \frac{355}{113}\right\}$.

Crossing pi

Theorem 2 (Crossover mediant). *Suppose that* $\left\{\frac{a}{b},\frac{c}{d}\right\}$, *is a bracket of* $\pi-$ *approximants, with* $b,d \in \mathbb{N}$; $a,c \in \mathbb{N}_0$ *(whole numbers); and* $0<b<d$, *where* $a \stackrel{\text{def}}{=} Round[b\pi]$ *and* $c \stackrel{\text{def}}{=} Round[d\pi]$. *Then there exist a positive integer* k *and its associated monotonic sequence of* i^{th}-mediants $\left\{i\odot\frac{a}{b}\oplus\frac{c}{d} \mid i=0,1,\cdots,k\right\}$ *formed from* $\pi-$*approximants, where all but the last element, the* k^{th}-mediant, *lies on the* $\frac{c}{d}$ *-side of* π. *The* k^{th} -mediant $k\odot\frac{a}{b}\oplus\frac{c}{d}$ *is the* ***crossover mediant*** *and its associated index* k *is the* ***crossover index*** *(forming the first element on the* $^{a}/_{b}$ *-side of* π*), defined as:*

$$k \stackrel{\text{def}}{=} Ceiling\,[\kappa], \text{ where} \tag{2}$$

$$\kappa \stackrel{\text{def}}{=} \frac{d\pi-c}{a-b\pi} \stackrel{\text{def}}{=} \frac{d}{b}\left(\frac{\pi-c/d}{a/b-\pi}\right) \tag{3}$$

Proof. See Supplementary Section. Note: in the proof, a rearrangement is useful:

$$\kappa \odot \frac{a}{b} \oplus \frac{c}{d} = \frac{\kappa a + c}{\kappa b + d} = \pi, \tag{4}$$

Which elements from this sequence of ith-mediants make the gold list?

Remark. *In Theorem 4, we will address including the kth-mediant in the gold list. We always include the $(k-1)^{st}$-mediant, except for the special case where $k = 1$. For then the $(k-1)^{st}$-mediant (0th) is $\frac{c}{d}$, already considered for the gold list.*

Theorem 2 shows that each pre-crossover i^{th}-mediant $(1 \le i \le k-1)$ is a π-approximant, closer to π than its predecessor and closer ***than*** $\frac{\boldsymbol{c}}{\boldsymbol{d}}$. But no guarantee exists that any mediants except the $(k-1)^{st}$ are closer to c ***than is*** $\frac{\boldsymbol{a}}{\boldsymbol{b}}$. Consequently, these earlier i^{th}-mediants may not make the gold list. Post-crossover i^{th}-mediants $(i > k)$ might not be π-approximants at all. Their progenitors are fractions on the same side of π, where Theorem 1 does not apply.

We now resolve the "closer **than** $^{a}/_{b}$" question.

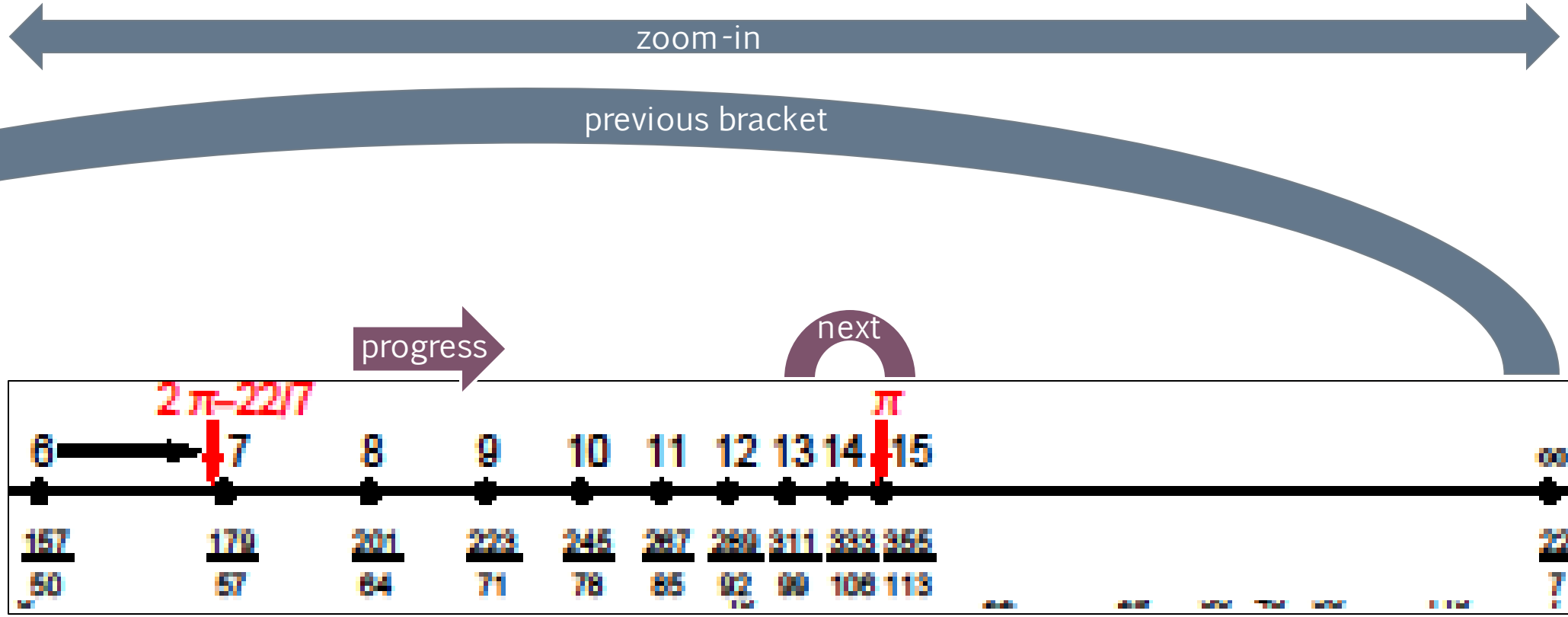
Corollary 1 (Gold mediant). *Suppose that the conditions in Theorem 2 hold. In that case, there exist a positive integer* j *and its subsequence of* i^{th}*-mediants, namely* $\left\{i\odot\frac{a}{b}\oplus\frac{c}{d}\,|i=j-1,j,\cdots,k-1,k\right\}$, *where, except for the first mediant, the* $(j-1)^{st}$, *all the rest are closer to* π *than is* $\frac{a}{b}$. *The* j^{th} *-mediant* $j\odot\frac{a}{b}\oplus\frac{c}{d}$ *is the* ***gold mediant*** *and the index* j *is the* ***gold index*** *(forming the first new gold-list element), defined as:*

$$j \stackrel{\text{def}}{=} Max\left[1, Ceiling[\eta]\right], \text{ where } \eta \stackrel{\text{def}}{=} \frac{1}{2}\left(\kappa-\frac{d}{b}\right) \tag{5}$$

Proof. See Supplementary Section.

There, we make use of the Ballew extension [3], that the sum of *explementary* pairs is 2π. (This is a natural extension of supplementary pairs with sum π and complementary pairs with sum π/2.)

Remark. *If* $j = 0$, *then the* 0th *-mediant is already closer to* π than is $^{a}/_{b}$. *Regardless, all further* j^{th} *-mediants* $(i = j, j+1, \cdots, k-1)$ *will be still closer to* π *and included in the gold list. Readers can verify that Equation (5) gives* $j = 7$ *for the weighted-mediant sequence in Figure 10. So the* 7th *-mediant* $\frac{179}{57}$ (*of bracket* $\left\{\frac{22}{7}, \frac{25}{8}\right\}$) *is the first new element included in the gold list, the first whose distance from* π *is less than the distance between* $\frac{22}{7}$ *and* π.



Zoom: *Equation (5) gives* $j = 7$, so the 7th-mediant, $\frac{179}{57}$, is the first one closer closer to π than $2\pi - 22/7$.

$2\pi - 22/7$ is the explement of $22/7$ on the opposite side of π

Figure 10 (Figured 9, reconsidered).

π

Summary. The sequence of ith -mediants $\left\{\mathrm{i}\odot\frac{\mathrm{a}}{\mathrm{b}}\oplus\frac{\mathrm{c}}{\mathrm{d}}\mid \mathrm{i}=\mathrm{j},\mathrm{j}+1,\cdots,\mathrm{k}-1\right\}$ approaches π monotonically:

- $k \stackrel{\text{def}}{=} Ceiling\,[\kappa]$, *where* $\kappa \stackrel{\text{def}}{=} \frac{d\pi-c}{a-b\pi} = \frac{d}{b}\left(\frac{\pi-c/d}{a/b-\pi}\right)$
- $j \stackrel{\text{def}}{=} Max\left[1, Ceiling[\eta]\right]$, *where* $\eta \stackrel{\text{def}}{=} \frac{1}{2}\left(\kappa-\frac{d}{b}\right)$

Its elements must all appear in the gold list; the kth-mediant might not.

Also, from the bracket $\left\{\frac{a}{b},\ \frac{c}{d}\right\}$ of π-approximants, we construct a better bracket $\left\{(k-1)\odot\frac{a}{b}\oplus\frac{c}{d}, k\odot\frac{a}{b}\oplus\frac{c}{d}\right\}$, and then iterate.

Iterating brackets – 5th (final) method

Definition 4. *The **recursive fractions*** $\frac{a_{m+1}}{b_{m+1}}$ *and* $\frac{c_{m+1}}{d_{m+1}}$ *are now defined in four parts:*

$$\begin{aligned} &\blacksquare\ a_{m+1} \stackrel{\text{def}}{=} (k-1)a_m + c_m && \blacksquare\ c_{m+1} \stackrel{\text{def}}{=} ka_m + c_m \\ &\blacksquare\ b_{m+1} \stackrel{\text{def}}{=} (k-1)b_m + d_m && \blacksquare\ d_{m+1} \stackrel{\text{def}}{=} kb_m + d_m, \end{aligned} \tag{6}$$

because $\frac{a_{m+1}}{b_{m+1}} \stackrel{\text{def}}{=} (k-1)\odot\frac{a_m}{b_m}\oplus\frac{c_m}{d_m}$ *and* $\frac{c_{m+1}}{d_{m+1}} \stackrel{\text{def}}{=} k\odot\frac{a_m}{b_m}\oplus\frac{c_m}{d_m}$.

Theorem 3 - Iterative bracket

Theorem 3 (Iterative bracket). *Suppose that* $\left\{\frac{a}{b}, \frac{c}{d}\right\}$ *is a bracket of* π*-approximants. Then each step:*

1) generates the next iterative bracket for π

2) reverses the order of its endpoints

3) reduces its length

If $|ad - bc| = 1$*, or equivalently, the bracket length is* $\left|\frac{a}{b} - \frac{c}{d}\right| = \frac{1}{bd}$*, this length is less than half that of the previous interval.*

Claim 1 (Generates). *By iteration,*

$$\left\{(k-1)\odot\frac{a}{b}\oplus\frac{c}{d},\ k\odot\frac{a}{b}\oplus\frac{c}{d}\right\} \text{ is also a bracket of } \pi\text{-approximants.}$$

Proof. From Theorem 1, $(k-1)\odot\frac{a}{b}\oplus\frac{c}{d}$ and $k\odot\frac{a}{b}\oplus\frac{c}{d}$ are opposite-sided π-approximants. The next generated bracket is $\left\{(k-1)\odot\frac{a}{b}\oplus\frac{c}{d},\ k\odot\frac{a}{b}\oplus\frac{c}{d}\right\}$ in this order, since $(k-1)\odot\frac{a}{b}\oplus\frac{c}{d}$ is the earlier fraction with the smaller denominator. ■

Claim 2 (Reverses). *Exactly one equivalence is true:*

$$\begin{aligned} &\textit{either } \frac{a}{b}<\pi<\frac{c}{d} \textit{ if and only if } (k-1)\odot\frac{a}{b}\oplus\frac{c}{d}>\pi>k\odot\frac{a}{b}\oplus\frac{c}{d} \\ &\textit{or } \quad \frac{a}{b}>\pi>\frac{c}{d} \textit{ if and only if } (k-1)\odot\frac{a}{b}\oplus\frac{c}{d}<\pi<k\odot\frac{a}{b}\oplus\frac{c}{d} \end{aligned} \tag{7}$$

Proof. Use *k – 1* as *v* and use *k* as *w* in Inequalities (1). ■

Claim 3 (Reduces). *Bracket lengths become smaller from parent* $\left\{\frac{a}{b}, \frac{c}{d}\right\}$ *to child* $\left\{(k-1)\odot\frac{a}{b}\oplus\frac{c}{d},\ k\odot\frac{a}{b}\oplus\frac{c}{d}\right\}$:

$$\left|k\odot\frac{a}{b}\oplus\frac{c}{d}-(k-1)\odot\frac{a}{b}\oplus\frac{c}{d}\right|<\left|\frac{a}{b}-\frac{c}{d}\right|.$$

In the special case in which bracket elements satisfy the property $|ad-bc|=1$:

$$\left|\frac{a}{b}-\frac{c}{d}\right|=\left|\frac{ad-bc}{bd}\right|=\frac{1}{bd} \tag{8}$$

and every child bracket inherits Equation (8), with less that half its parent's length.

Proof. Using mathematical induction. See Supplementary Section.

Remark. *The minimum bracket-length ratio (marginally greater than 2) occurs when* $k = 1$. *At the other extreme, if* $k \gg 1$, *with* $b \approx d$, *we get a much different ratio:*

$$\frac{current\ bracket\ length}{next\ bracket\ length} = \frac{current\left(\frac{1}{bd}\right)}{next\left(\frac{1}{bd}\right)} \approx ((k-1)+1)(k+1) = order(k^2)$$

Iterative brackets provide a faster way to approach π and "magically" replicate the gold list. Next, we need a definitive way to choose which bracket element to include in our tracked fractions, the earlier or later one.

Choosing sides – the better bracket element

Property 3 (Recursive κ). *Here is an equivalent, iterative method for finding each κ from its predecessor:*

$$\kappa_{m+1} = \frac{\mathbf{k_m - \kappa_m}}{\mathbf{\kappa_m - (k_m - 1)}} = \frac{\mathbf{Ceiling[\kappa_m] - \kappa_m}}{\mathbf{\kappa_m - Floor[\kappa_m]}} = \frac{1}{\mathrm{frac}[\kappa_m]} - 1 \qquad (9)$$

From the two middle expressions in bold, a graphic highlights the relative position of κ_m between $k_m = Ceiling[\kappa_m]$ and $k_m - 1 = Floor[\kappa_m]$, an emphasis of our research:

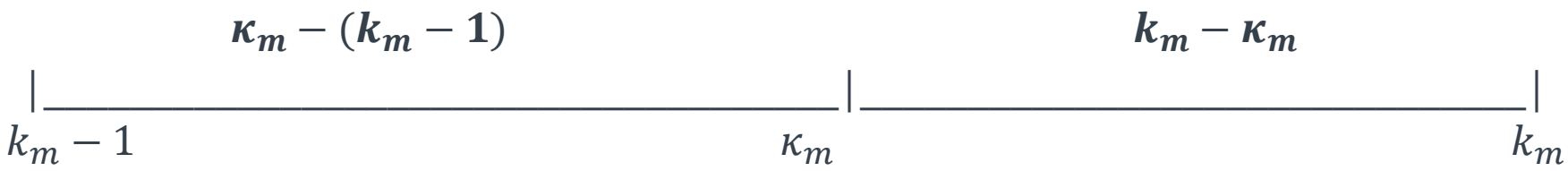


Figure 11.

Derivation. See Supplementary Section.

We now use κ to determine if the earlier or later fraction is closer to π.

Theorem 4 (Mediant choice). *Suppose that* $\left\{\frac{a_m}{b_m}, \frac{c_m}{d_m}\right\}$ *is a bracket of* π*-approximants. As usual, define* $k_m \stackrel{\text{def}}{=} Ceiling[\kappa_m]$. *The* k_m^{th} *-mediant must be included in the gold list if either equivalent statement for* $m+1$ *or for* m *is true*:

$$\text{If} \qquad \kappa_{m+1} < \frac{d_{m+1}}{b_{m+1}} \tag{10}$$

$$\text{or} \qquad \frac{k_m - \kappa_m}{\kappa_m - (k_m - 1)} < \frac{k_m + \frac{d_m}{b_m}}{(k_m - 1) + \frac{d_m}{b_m}} = \frac{1}{2}\left(\frac{k_m + \frac{d_m}{b_m}}{\left(k_m - \frac{1}{2}\right) + \frac{d_m}{b_m}}\right) \tag{11}$$

Proof. See Supplementary Section.

Remark. *Inequality (10) provides the simplest test of whether the* k_m^{th} *-mediant belongs in the gold list. Theorems 2 and 4 jointly specify which new bracket elements are in that list.*

Concluding

On the journey to find better π–approximants, either take the express lane (best iterative-bracket approximant) or follow the scenic route (entire gold list).

In the express lane, simply generate the sequence of brackets, each with its crossover index k. Theorem 4 determines which endpoint to use as the approximant "best so far." The lower limit for convergence toward π is a factor of two per iteration; in practice, the improvement averages greater than ten per step.

On the scenic route, calculate for each bracket both indices, gold j and crossover k. Include all mediants from j to $k-1$, and k too, if Theorem 4 is satisfied.

ITERATIVE BRACKETS

Iterative-bracket – 5th method

- Initial π-approximants, using the iterative-bracket /mediant-choice method
- Gold step# comparison

Table 6.

5th	Gold	better n/d choice	digits	error
01	02	$\frac{13}{4}$	1	+1.08E-01
02	05	$\frac{22}{7}$	3	+1.26E-03
03	14	$\frac{355}{113}$	7	+2.67E-07
05	162	$\frac{104\,348}{33\,215}$	10	+3.32E-10
06	163	$\frac{208\,341}{66\,317}$	10	-1.22E-10
07	164	$\frac{312\,689}{99\,532}$	11	+2.91E-11
09	166	$\frac{1\,146\,408}{364\,913}$	12	+1.61E-12
11	169	$\frac{5\,419\,351}{1\,725\,033}$	14	+2.21E-14
13	177	$\frac{80\,143\,857}{25\,510\,582}$	15	-5.79E-16
14	179	$\frac{245\,850\,922}{78\,256\,779}$	16	-7.82E-17
15	180	$\frac{411\,557\,987}{131\,002\,976}$	17	+1.94E-17

Advantages in time and iterations

The computational advantage of the new algorithm over its predecessor is staggering. Initially, we compared times and iterations to reach 208,341/66,317 (10 correct π-digits). A core i5 (2.0 GHz) computer required 8.64 seconds CPU time to construct the first 163 approximants of the gold list, using the *enhanced increment-denominator* (3rd) method. By contrast, the *iterative-bracket* (5th) algorithm was over 10,000 times faster, needing only 0.836 millisecond and 5 iterations to reach the same point.

In 1/32 second and only 80 steps, we generate a rational approximation to π, with a 40-digit numerator and denominator, accurate to 80 digits:

$$\frac{4,427,007,044,615,115,050,134,854,648,525,685,871,587}{1,409,160,108,506,276,783,085,718,440,252,375,099,653}$$

Table 7 lists estimates with mnemonic potential. This now completes our journey to find rapidly more accurate and memorable rational approximations to irrational π. Still, we must memorize forty numerator plus forty denominator digits to get eighty digits of π. So McLoone had a point, when he blogged, “All rational approximations of pi are useless.”

Mnemonic π-approximants

digits	n / d
7	3 55 / 11 3
9	103 99 3 / 33 104
14	541 9 351 / 17 25 0 33
16	1 65 70 70 65 / 5 2 7 46 19 7
17	4 11 55 798 7 / 1 31 00 29 76
18	10 6896 6896 / 34 0 26 27 31
26	895 8937 76 8937 / 285 17 18 46 15 58
33	666 27 4455 92 88887 / 21 20 81 74 62 33 89 16

Table 7.

Estimates with mnemonic potential (our secondary goal), mined from the gold list.

π

Extending

So far, we only assumed the irrationality of π. Using the same strategy on other irrationals, we can also find their rational approximants, with about the same combined number of digits as those correct in their decimal representations.

A crucial first step is seeding the sequence effectively. A good initial bracket $\left\{\frac{a}{b}, \frac{c}{d}\right\}$ meets the condition $ad - bc = \pm 1$. If the condition is not met at first, generated brackets seem to comply later, an empirical result worth provoking further study. Table 8 lists approximants for a variety of irrationals and rationals.

Approximants for:

- six irrationals
- a repeating decimal
- a terminating decimal

#	digits	n/d
e	10	49,171/18,089
e^{π}	8	10,691/4621
$e^{-\pi/2}$	8	649/3122
$\sqrt[12]{2}$	9	18,904/17,843
$\varphi \stackrel{\text{def}}{=} ½(1+\sqrt{5})\ (golden\ ratio)$	6	1597/987
$\gamma\ (Euler's\ Constant)$	12	323,007/559,595
$1.\overline{765432}\cdots$	all	196,159/111,111
$(146{,}097\text{d}/400\text{y}) \div (7d/wk)^*$	all	20,871/400 wk/yr

*Number of weeks in a year, for the Gregorian calendar

Table 8.

We can even use the iterative-bracket algorithm on rationals to generate their (finite) gold list, with the target itself, reduced to lowest terms, as the final "approximant." Here is the gold list for the last entry in Table 7:

$$\left\{\frac{52}{1}, \frac{157}{3}, \frac{209}{4}, \frac{261}{5}, \frac{313}{6}, \frac{574}{11}, \frac{887}{17}, \frac{2{,}348}{45}, \frac{3{,}235}{62}, \frac{5{,}583}{107}, \frac{8{,}818}{169}, \frac{12{,}053}{231}, \frac{20{,}871}{400}\right\}$$

These fractions represent increasingly accurate approximations to the number of weeks in a calendar year, $52.1775 = 52\frac{71}{400}$, using the Gregorian leap-year definition.

Supplementary material in the Appendix

- Proofs / derivations, including treatment of the rational target case
- Advantages / comparisons:
  - Continued fractions
  - Greek ladders
  - Logarithmic approximations
- *Mathematica* code for the iterative-brackets method
- References

IN MEMORIAM

The authors wish to recognize Tom M. Apostol (Caltech) and Robert J. Wisner (New Mexico University) for their extensive and significant contributions to mathematics during their lifetimes and for their obvious love of mathematics, inspiring listeners and readers alike.

The End

Appendix

Property 1 (Mediant inequality).

Derivation. Both needed inequalities are true:

Left: $\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc-ad}{b(b+d)} = \frac{c}{b+d} - \frac{ad}{b(b+d)} = \frac{d}{b+d}\left(\frac{\boldsymbol{c}}{\boldsymbol{d}} - \frac{\boldsymbol{a}}{\boldsymbol{b}}\right) > \mathbf{0}$

Right: $\frac{c}{d} - \frac{a+c}{b+d} = \frac{bc-ad}{d(b+d)} = \cdots = \frac{b}{b+d}\left(\frac{\boldsymbol{c}}{\boldsymbol{d}} - \frac{\boldsymbol{a}}{\boldsymbol{b}}\right) > \mathbf{0}$

For $\left(\frac{c}{d} - \frac{a}{b}\right)$ is assumed positive. ■

Theorem 1 (Mediant of approximants).

Proof. Let a/b and c/d be π-approximants; i.e., each numerator is the best choice for that denominator. For fractions on opposite sides of π, we need only consider:

$$\frac{a}{b} < \pi < \frac{c}{d} \tag{12}$$

That a/b is a π-approximant means $a \stackrel{\text{def}}{=} Round\,[b\,\pi]$. By construction,

$$\text{then } -0.5 < b\pi - a < 0.5,$$

$$\text{so } \quad a - 0.5 < b\pi < a + 0.5, \text{ and similarly, } c - 0.5 < d\pi < c + 0.5. \tag{13}$$

Assumption (12) gives $a < b\pi$ and $d\pi < c$, strengthening Inequalities (13):

$$a - 0.5 < \boxed{a < b\pi < a + 0.5}, \text{ and } \boxed{c - 0.5 < d\pi < c} < c + 0.5$$

Adding their corresponding terms

$$\text{results in} \qquad (a + c) - 0.5 < (b + d)\pi < (a + c) + 0.5,$$

$$\text{and then} \qquad -0.5 < (b + d)\pi - (a + c) < 0.5.$$

So $(a + c) = Round[(b + d)\pi]$, meaning that $\frac{a+c}{b+d}$ is also a π-approximant. ■

Mediant inequality, three versions

Definition 3. *The **weighted mediant** of fractions a/b and c/d is (all versions)*

- $w \odot \frac{a}{b} \oplus \frac{c}{d} \stackrel{\text{def}}{=} \frac{wa + c}{wb + d}$ — BRIEF FORM
- $(w, 1) \odot \frac{a}{b} \oplus \frac{c}{d} \stackrel{\text{def}}{=} \frac{wa+c}{wb+d}$ — FULL FORM
- $(1, z) \odot \frac{a}{b} \oplus \frac{c}{d} \stackrel{\text{def}}{=} \frac{a+wc}{b+wd}$ — LATER FRACTION
- $(w, z) \odot \frac{a}{b} \oplus \frac{c}{d} \stackrel{\text{def}}{=} \frac{wa+zc}{wb+zd}$ — BOTH FRACTIONS

Property 2 (Weighted mediant inequality).

Derivation. Consider the case where $\frac{a}{b} < \frac{c}{d}$, meaning $bc - ad > 0$:

$$v < w \text{ implies that } v(bc - ad) < w(bc - ad),$$

$$\text{leading to } (vb + d)(wa + c) < (wb + c)(va + c).$$

So $\frac{wa + c}{wb + d} < \frac{va + c}{vb + d}$, implying $\frac{a}{b} < w \odot \frac{a}{b} \oplus \frac{c}{d} < v \odot \frac{a}{b} \oplus \frac{c}{d} \leq \frac{c}{d}$.

For the $\frac{a}{b} > \frac{c}{d}$ case, multiplying by $(bc - ad)$ reverses all inequalities after $v < w$.

$$\text{Therefore, } \frac{a}{b} > w \odot \frac{a}{b} \oplus \frac{c}{d} > v \odot \frac{a}{b} \oplus \frac{c}{d} \geq \frac{c}{d}.$$

■

Theorem 2 (Crossover mediant).

Proof. The result follows from Definition 3 and Theorem 1. We show that the sequence is monotonic, with each element a π-approximant.

- For $i = 0$: the zero element $0 \odot \frac{a}{b} \oplus \frac{c}{d} = \frac{0a + c}{0b + d} = \frac{c}{d}$, naturally on its side of π.
- For $i = 1$: the 1st-mediant $1 \odot \frac{a}{b} \oplus \frac{c}{d} = \frac{a}{b} \oplus \frac{c}{d}$ is the simple mediant. With opposite-sided progenitors, Theorem 1 ensures that their mediant is also a π-approximant in $\{a/b, c/d\}$. If it is still on the $\frac{c}{d}$-side of π, Theorem 1 then guarantees that the 2nd-mediant $2 \odot \frac{a}{b} \oplus \frac{c}{d} = \frac{a}{b} \oplus \frac{a}{b} \oplus \frac{c}{d}$ is likewise a π-approximant.

π

- Incrementing i leads to the kth-mediant, first on the $\frac{a}{b}$-side of π, existing since:

$$\lim_{i\to\infty} i\odot\frac{a}{b}\oplus\frac{c}{d} = \lim_{i\to\infty}\frac{ia + c}{ib + d} = \frac{a}{b}$$

- By finitely repeated applications of Theorem 1, for all $i \in \{1, \cdots, k\}$, each ith-mediant $i\odot\frac{\mathrm{a}}{\mathrm{b}}\oplus\frac{\mathrm{c}}{\mathrm{d}} = \frac{\mathrm{a}}{\mathrm{b}}\oplus(i-1)\odot\frac{\mathrm{a}}{\mathrm{b}}\oplus\frac{\mathrm{c}}{\mathrm{d}}$ is a π-approximant too. So, all but the kth-mediant are necessarily closer to a/b than their predecessors. Therefore, the sequence of ith-mediants monotonically approaches $\frac{a}{b}$, and except for the final element, monotonically approaches π.

Here the crossover index k is the smallest integer placing the kth-mediant between π and $\frac{a}{b}$. The previous $(k-1)^{st}$-mediant was still between $\frac{c}{d}$ and π. Clearly, both are closer to π, on opposite sides. So $k \stackrel{\text{def}}{=} Ceiling[\kappa]$, where κ is the irrational solution to the relation $\kappa \odot \frac{a}{b} \oplus \frac{c}{d} = \frac{\kappa a + c}{\kappa b + d} = \pi$. A closed-form expression can now be obtained, providing an equivalent definition for κ in Equation (2):

$$\text{Rearranging } \frac{\kappa a + c}{\kappa b + d} = \pi \text{ gives } \kappa \stackrel{\text{def}}{=} \frac{d\pi - c}{a - b\pi} \stackrel{\text{def}}{=} \frac{d}{b}\left(\frac{\pi - c/d}{a/b - \pi}\right).$$

The ratio in parentheses $\left(\frac{\pi - c/d}{a/b - \pi}\right)$ must be positive. The numerator and denominator are either both positive or both negative, because π lies between $\frac{a}{b}$ and $\frac{c}{d}$. So κ is a positive real number, making $k \stackrel{\text{def}}{=} Ceiling[\kappa]$ a positive integer, which identifies the first mediant on the $\frac{a}{b}$-side, the kth-mediant, in Equation (2):

$$k \stackrel{\text{def}}{=} Ceiling\,[\kappa],\ \textit{where}\ \kappa \stackrel{\text{def}}{=} \frac{\pi d - c}{a - \pi b} \stackrel{\text{def}}{=} \frac{d}{b}\left(\frac{\pi - c/d}{a/b - \pi}\right) \qquad \blacksquare$$

Corollary 1 (Gold mediant).

Proof. By the Ballew extension [3], the sum of *explementary* pairs is 2π. (Supplementary pairs have sum π; complementary pairs have sum π/2.) The *explement* of $\frac{a}{b}$ *is* $\left(\frac{a}{b}\right)' \stackrel{\text{def}}{=} 2\pi - \frac{a}{b}$, or rearranged, $\pi - \left(\frac{a}{b}\right)' = \frac{a}{b} - \pi$. So $\left(\frac{a}{b}\right)'$ is the same distance from π as $\frac{a}{b}$, but on the $\frac{c}{d}$-side of π.

We consider the case where $\frac{c}{d} < \frac{a}{b}$. (The result is identical, if $\frac{a}{b} < \frac{c}{d}$.) For the jth-mediant to be *the first one closer* to π than is $\frac{a}{b}$, the explement $\left(\frac{a}{b}\right)'$ must lie between the $(j-1)^{st}$ and j^{th}-mediant:

$$(j-1)\odot\frac{a}{b}\oplus\frac{c}{d} < \left(\frac{a}{b}\right)' < j\odot\frac{a}{b}\oplus\frac{c}{d}$$

Following the "k and κ" analogy as a road map, we define j and η implicitly:

$$j \stackrel{\text{def}}{=} Ceiling[\eta], \text{ where } \eta \odot \frac{a}{b} \oplus \frac{c}{d} \stackrel{\text{def}}{=} \frac{\eta a + c}{\eta b + d} \stackrel{\text{def}}{=} 2\pi - \frac{a}{b} = \left(\frac{a}{b}\right)'$$

Solving for η (and reformulating one term as κ) yields a concise formula:

$$\text{Because} \quad \eta a + c = \eta\left(2\pi - \frac{a}{b}\right) b + \left(2\pi - \frac{a}{b}\right) d,$$

$$\text{then} \quad \eta\left(a - \left(2\pi - \frac{a}{b}\right) b\right) = \left(2\pi - \frac{a}{b}\right) d - c.$$

$$\text{So } \eta = \frac{b}{b}\left(\frac{\left(2\pi - \frac{a}{b}\right)d - c}{2(a - b\pi)}\right) = \frac{1}{2}\left(\frac{(bd\pi - bc) - (ad - bd\pi)}{b(a - b\pi)}\right) = \frac{1}{2}\left(\frac{\cancel{b}(\pi d - c)}{\cancel{b}(a - \pi b)} - \frac{d\cancel{(a - \pi b)}}{b\cancel{(a - \pi b)}}\right) \stackrel{\text{def}}{=} \frac{1}{2}\left(\kappa - \frac{d}{b}\right).$$

Therefore, we first ensure that the jth-mediant is closer to π **than is** $\frac{a}{b}$:

First condition: $j \stackrel{\text{def}}{=} Ceiling[\eta] \stackrel{\text{def}}{=} Ceiling\left[\frac{1}{2}\left(\kappa - \frac{d}{b}\right)\right]$.

Also, we require the jth-mediant to be closer to π **than is** $\frac{c}{d}$:

$$\frac{c}{d} = 0 \odot \frac{a}{b} \oplus \frac{c}{d} < j \odot \frac{a}{b} \oplus \frac{c}{d} = \frac{ja + c}{jb + d}.$$

Because $\frac{a}{b} > \frac{c}{d}$, this necessitates that the integer $j > 0$, that is:

Second condition: $j \geq 1$.

These results combine to give us Equation (4):

Joint conditions: $j \stackrel{\text{def}}{=} Max\left[1, Ceiling[\eta]\right]$, *where* $\eta \stackrel{\text{def}}{=} \frac{1}{2}\left(\kappa - \frac{d}{b}\right)$ ■

Theorem 3 (Iterative bracket), Claim 3 (Reduces).

Proof: By construction, the next bracket uses consecutive weighted mediants of the previous bracket $\left\{\frac{a}{b}, \frac{c}{d}\right\}$. Combining Relations (1) and (5):

$$\text{either} \quad \frac{a}{b} < k \odot \frac{a}{b} \oplus \frac{c}{d} < \pi < (k-1) \odot \frac{a}{b} \oplus \frac{c}{d} \leq \frac{c}{d}$$

$$\text{or} \quad \frac{c}{d} \leq k \odot \frac{a}{b} \oplus \frac{c}{d} < \pi < (k-1) \odot \frac{a}{b} \oplus \frac{c}{d} < \frac{a}{b}$$

Clearly, at least one child bracket element will be different from its parent elements and strictly between them. So the new bracket has a smaller length.

In the special case, we will prove by induction that the length of bracket $\{{}^{a}/_{b}, {}^{c}/_{d}\}$ satisfies Equation (7), $\left|\frac{a}{b}-\frac{c}{d}\right|=\frac{1}{bd}$. Initially, this equation is true when $n=1$, since $\left|\frac{3}{1}-\frac{13}{4}\right|=\frac{1}{1\times4}$. Assume that the equation is true when $n=m$, i.e., $\left|\frac{a_m}{b_m}-\frac{c_m}{d_m}\right|=\frac{|a_m d_m-b_m c_m|}{b_m d_m}=\frac{1}{b_m d_m}$, so $|a_m d_m-b_m c_m|=1$. From Equations (5), the bracket length $\left|\frac{c_{m+1}}{d_{m+1}}-\frac{a_{m+1}}{b_{m+1}}\right|$ is $\left|k_m\odot\frac{a_m}{b_m}\oplus\frac{c_m}{d_m}-(k-1)\odot\frac{a_m}{b_m}\oplus\frac{c_m}{d_m}\right|$

$=\left|\frac{k_m a_m+c_m}{k_m b_m+d_m}-\frac{(k_m a_m+c_m)-a_m}{(k_m b_m+d_m)-b_m}\right|=\frac{|a_m d_m-b_m c_m|}{b_{m+1}d_{m+1}}=\frac{1}{b_{m+1}d_{m+1}}$,

finishing the induction proof.

Using Equations (5) and (7), here is the ratio of bracket lengths:

$$\frac{current\left(\frac{1}{bd}\right)}{next\left(\frac{1}{bd}\right)} = \frac{((k-1)b+d)(kb+d)}{bd}$$

$$\frac{k(k-1)b^2+(k-1)bd+(kbd+d^2)}{bd} > 0+0+(k+1) > 2$$

(By assumption and construction, $k \geq 1$ and $\frac{d}{b} > 1$.)

This completes the proof of Claim 3. Thus, Theorem 3 is also proven. ■

Property 3 (Recursive kappa).

Derivation. Use the current bracket $\left\{\frac{a}{b}, \frac{c}{d}\right\}$ in Equations (2) and (3). This allows us to find the *next* κ in terms of the *current* κ and derive Equation (8):

$$\kappa_{m+1} \stackrel{\text{def}}{=} \frac{\pi d_{m+1} - c_{m+1}}{a_{m+1} - \pi b_{m+1}} = \frac{\left(\frac{\kappa a + c}{\kappa b + d}\right)(kb + d) - (ka + c)}{((k-1)a + c) - \left(\frac{\kappa a + c}{\kappa b + d}\right)((k-1)b + d)}$$

$$= \frac{(\kappa a + c)(kb + d) - (\kappa b + d)(ka + c)}{(\kappa b + d)((k-1)a + c) - (\kappa a + c)((k-1)b + d)} = \frac{(k - \kappa)\cancel{(bc - ad)}}{(\kappa - (k-1))\cancel{(bc - ad)}}$$

$$= \frac{\boldsymbol{k_m - \kappa_m}}{\boldsymbol{\kappa_m - (k_m - 1)}} = \frac{\mathbf{Ceiling}[\kappa_m] - \kappa_m}{\kappa_m - \mathbf{Floor}[\kappa_m]} = \frac{1}{\text{frac}[\kappa_m]} - 1 \qquad \blacksquare$$

(The expression $\frac{1}{\text{frac}[\kappa_m]} - 1$ is related to the continued fraction algorithm.)

Theorem 4 (Mediant choice).

Proof. First, assume that $\frac{a_{m+1}}{b_{m+1}} < \pi < \frac{c_{m+1}}{d_{m+1}}$, ensuring that $a_{m+1}d_{m+1} < b_{m+1}c_{m+1}$. Now use Equation (4) to describe π in terms of the $(m+1)^{st}$-iteration:

$$\pi = \frac{\kappa_{m+1}a_{m+1} + c_{m+1}}{\kappa_{m+1}b_{m+1} + d_{m+1}}$$

To include the next k_{m+1}^{st}-mediant, $\frac{c_{m+1}}{d_{m+1}}$ must be closer to π than is $\frac{a_{m+1}}{b_{m+1}}$ (temporarily dropping $(m+1)$ subscripts):

π

So $\frac{c_{m+1}}{d_{m+1}} - \pi < \pi - \frac{a_{m+1}}{b_{m+1}}$,

meaning $\frac{c}{d} - \frac{\kappa a + c}{\kappa b + d} < \frac{\kappa a + c}{\kappa b + d} - \frac{a}{b}$,

implying $\frac{\kappa(\cancel{bc - ad})}{d(\cancel{\kappa b + d})} < \frac{1(\cancel{bc - ad})}{b(\cancel{\kappa b + d})}$,

giving Inequality (10), $\kappa_{m+1} < \frac{d_{m+1}}{b_{m+1}}$, because $(bc - ad) > 0$.

Similarly, $\frac{a_{m+1}}{b_{m+1}} > \pi > \frac{c_{m+1}}{d_{m+1}}$ also leads to Inequality (10). This can be rewritten for m by Equations (5) and (8):

$$\frac{k - \kappa}{\kappa - (k-1)} < \frac{k + \frac{d}{b}}{(k-1) + \frac{d}{b}} = {}^1\!/_2 \left(\frac{k + \frac{d}{b}}{\left(k - \frac{1}{2}\right) + \frac{d}{b}} \right)$$ (without m subscripts) ■

The last part of (11) is derived by substitution. Interval $(k-1, k)$ has midpoint $k - \frac{1}{2}$.

Bracket Method with Rational Target

In rational cases, the bracket method will produce an integer value for κ when the bracket's later element equals the exact fraction and the iteration then terminates.

Derivation. For rational target T, κ becomes $\frac{dT-c}{a-bT}$, also rational. Assume that κ_m is an **integer** for some m. Then $k_m = Ceiling[\kappa_m] = \kappa_m$. Applying the still valid recurrence relation, Equation (9), $\kappa_{m+1} = \frac{k_m - \kappa_m}{k_m - (k_m - 1)} = \frac{0}{1} = 0$. But we chose κ_{m+1} so that $T = \kappa_{m+1} \odot \frac{a_{m+1}}{b_{m+1}} \oplus \frac{c_{m+1}}{d_{m+1}}$.

Therefore, $T = 0 \odot \frac{a_{m+1}}{b_{m+1}} \oplus \frac{c_{m+1}}{d_{m+1}} = \frac{c_{m+1}}{d_{m+1}}$.

Rational extension, part 2

If κ_m is a **noninteger** rational, the simplest form of the recurrence relation can find the next κ directly, from Equation (9), $\kappa_{m+1} = \frac{1}{frac[\kappa_m]} - 1$. But this is simply the Euclidean algorithm to find the (finite) continued fraction (CF) representation of a rational number, a well-known process that terminates with an integer!

Therefore, for a rational target, the sequence $\{\kappa_1, \kappa_2, \cdots, \kappa_m, 0\}$ always terminates (at some level m) with an integer κ_m and a final $\kappa_{m+1} = 0$. These signs indicate that the later element of the $(m+1)^{st}$-bracket is exactly equal to the target. ■

Calculated advantages in time and iterations

The computational advantage of the new algorithm over its predecessor is staggering. Initially, we compared times and iterations to reach 208,341/66,317 (10 correct π-digits). A core i5 (2.0 GHz) computer required 8.64 seconds CPU time to construct the first 163 approximants of the gold list, using the *enhanced increment-denominator* (3rd) method. By contrast, the *iterative-bracket* (5th) algorithm was over 10,000 times faster, needing only 0.836 millisecond and 5 iterations to reach the same point.

The iterative-bracket method exhibited roughly linear growth in the time needed for additional iterations. It required 1.89 millisecond and 18 iterations to reach the 185th gold-list entry---6,167,950,454/1,963,319,607 (19 digits). Meanwhile, the slower method needed 82,805 seconds (about 23 hours), a timings ratio of approximately 40,000:1.

Projected advantages

Suppose that this exponential increase in CPU time continues for the slower method. In 14 billion years, the computer finds only about 220 gold-list elements, a feat accomplished by the iterative-bracket method after 21 iterations in 2.02 *milliseconds!* Far closer approximants are within easy reach, using the faster method. In $1/32$ second and only 80 steps, we generate a rational approximation to π, with a 40-digit numerator and denominator, accurate to 80 digits:

$$\frac{4{,}427{,}007{,}044{,}615{,}115{,}050{,}134{,}854{,}648{,}525{,}685{,}871{,}587}{1{,}409{,}160{,}108{,}506{,}276{,}783{,}085{,}718{,}440{,}252{,}375{,}099{,}653}$$

So in 80 iterations, the bracket length enclosing π shrinks by 86 orders of magnitude, an average convergence rate of better than ten times (one decimal digit) per iteration.

Comparisons

Earlier published methods create a single approximation series, using recursive formulas to find the next numerator and denominator, then to track the lower and/or upper bounds. We find it simpler to create two related series of bracket elements, forming upper and lower bounds, both based on the single variable κ.

Our method has similarities and connections to previous work, including approximants generated from the continued fraction (CF) rendition of real numbers. Our own independent development came about because *"the data spoke to me,"* one of us said. We offer a decimal and visual approach related to, but not explicitly using CFs.

Continued fractions

Definition 5. *To form a **continued fraction** from any rational or irrational:*

- *Subtract the whole-number part as the first sequence term; append a semicolon.*
- *To attain the desired accuracy, repeatedly*
  - *take the reciprocal of the fractional part;*
  - *subtract the whole-number part as the next sequence term; append a comma.*

Similarities and differences

CONTINUED FRACTIONS (CF'S)

- Historical method to approximate irrationals
- In general real-number form
- x_{m+1} detailed in Definition 5
- a_n^{th}-mediant of previous two CF's

$$\frac{h_{n+1}}{k_{n+1}} \stackrel{\text{def}}{=} a_n \odot \frac{h_n}{k_n} \oplus \frac{h_{n-1}}{k_{n-1}}.$$

BRACKET METHOD

- New method
- In k-sequence form
- κ_{n+1} defined as $\frac{1}{frac[\kappa_m]} - 1$, Eq. (8)
- k_n^{th}-mediant of earlier/later fractions

$$\frac{c_{n+1}}{d_{n+1}} \stackrel{\text{def}}{=} k_n \odot \frac{a_n}{b_n} \oplus \frac{c_n}{d_n}.$$

CONTINUED FRACTIONS (CF'S)

- Rudimentary bracket

$$\left\{0\odot\left(\frac{a}{b}\oplus\frac{c}{d}\right),k\odot\left(\frac{a}{b}\oplus\frac{c}{d}\right)\right\}.$$

BRACKET METHOD

- Advanced bracket

$$\left\{(k-1)\odot\left(\frac{a}{b}\oplus\frac{c}{d}\right),k\odot\left(\frac{a}{b}\oplus\frac{c}{d}\right)\right\}.$$

- They produce similar fractions: alternately lower and higher than their target
- Similar convergence: moderate—algebraic irrationals, rapid—transcendentals
- For golden ratio φ: both give $k=1$ with all iterations, slowest convergence observed
- Bracket-length ratio is $\varphi+1\approx 2.6$: significantly larger than 2, the lower bound proven in Theorem 3.

Greek ladder methods

Herzinger/Wisner write about *Connecting Greek ladders and continued fractions*, while Wisner connects Greek ladders with Farey fractions and root approximations [5, 9, 10]. In an article, *Extending Theon's ladder to any square root* [4], Gilberson/Osler extend the ladder's scope from $\sqrt{2}$. They adapt recursive relations and skip rungs to converge faster, with definitions similar to ours. We extend approximations from π to any irrational, and we skip the optimal number of iterations.

Logarithmic approximations

In their article *Good rational approximations to logarithms* [1], Apostol/Mnatsakanian find fractional approximations for $\log_{10} k$, illustrated by $\log_{10} 2$. They use mediants that weight both lower and upper bounds:

$$\left\{\frac{3}{10}, \frac{1}{3}, \frac{\cancel{31}}{\cancel{103}}, \frac{28}{93}, \frac{\cancel{59}}{\cancel{196}}, \frac{146}{485}, \frac{\cancel{205}}{\cancel{681}}, \frac{\cancel{789}}{\cancel{2{,}621}}, \frac{643}{2{,}136}, \frac{4{,}004}{13{,}301}, \frac{\cancel{4{,}647}}{\cancel{15{,}437}}, \frac{12{,}655}{42{,}039}, \frac{8{,}651}{\cancel{28{,}738}}\right\}.$$

We instead weight only the earlier progenitor, using κ to pinpoint the crossover index and its mediant that has crossed over. In Table 9, each iterative step yields the next k in parallel with the next bracket, an approximant pair with at least one gold-list element. We discard six worse mediants, $\left\{\frac{\cancel{31}}{\cancel{103}}, \frac{\cancel{59}}{\cancel{196}}, \frac{\cancel{205}}{\cancel{681}}, \frac{\cancel{789}}{\cancel{2{,}621}}, \frac{\cancel{4{,}647}}{\cancel{15{,}437}}, \frac{\cancel{8{,}651}}{\cancel{28{,}738}}\right\}$, and include two better ones, $\left\{\frac{1}{2}, \frac{87}{289}\right\}$, a four-step improvement.

- Brackets of better (bold) approximants to $\mathbf{log_{10}2}$, using iterative-bracket method.
- More efficient/accurate than Apostol/Mnatsakanian mediant-bounds method (9 steps versus 13 steps).
- We discard from steps 4-9: ~~6 worse mediants~~
- Include from steps 2 & 5: [2 better mediants]
- Both omit: ~~1/4~~ and ~~4/13~~

Table 9.

$\mathrm{LOG}_{10}\,2$ APPROXIMATIONS

#	k	a/b	c/d	*digits*
01	2	~~(0/1)~~	[1/2]	1
02	3	1/3	~~1/4~~	1
03	9	3/10	~~4/13~~	3
04	2	28/93	~~31/103~~	4
05	2	~~59/196~~	[87/289]	5
06	4	146/485	~~205/681~~	6
07	6	643/2,136	~~789/2,621~~	7
08	2	4,004/13,301	~~4,647/15,437~~	9
09	1	~~8,651/28,738~~	12,655/42,039	9

Mathematica code — gold-list algorithm

```
MaxExtraPrecision = 1000; target = Pi (* change target and/or precision as needed *)
kList = jList = {}; a = Round[target]; b = 1; goldList = {a/b}; d = 2;

While[c = Round[target d];
   If[Abs[c/d - target] < Abs[Last[goldList] - target], AppendTo[goldList, c/d]];
   ! (a/b <= target <= c/d || a/b >= target >= c/d), d++];

brktList = {{a/b, c/d}};

(* main loop *)
While[Length[brktList] <= 15, (* or other convenient limit *)
   AppendTo[kList, k = Ceiling[κ = (d target - c)/(a - b target)]];
   If[k == 0, Break[]];
```

Mathematica code (concluded)

```
  AppendTo[jList, j = Max[1, Ceiling[(κ - d/b)/2]]];
   goldList = Join[goldList, Table[(i a + c)/(i b + d), {i, j, k - 1}]];
   If[(k - κ)/(κ - (k - 1)) < (k b + d)/((k - 1) b + d),
      AppendTo[goldList, (k a + c)/(k b + d)]];
   {a, b, c, d} = {(k - 1) a + c, (k - 1) b + d, k a + c, k b + d};
   AppendTo[brktList, {a/b, c/d}]];
(* end main loop *)

Print["Number of iterations = ", Length@kList, ", gold list length = ",
  Length@goldList];

Print@Grid[Transpose@MapThread[Prepend,{{Range[Length[brktList]],
   brktList,PadRight[jList,Length[brktList],"-"],
   PadRight[kList, Length[brktList], "-"]}, {"n:", "bracket:", "j:", "k:"}}]];

goldList
```

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