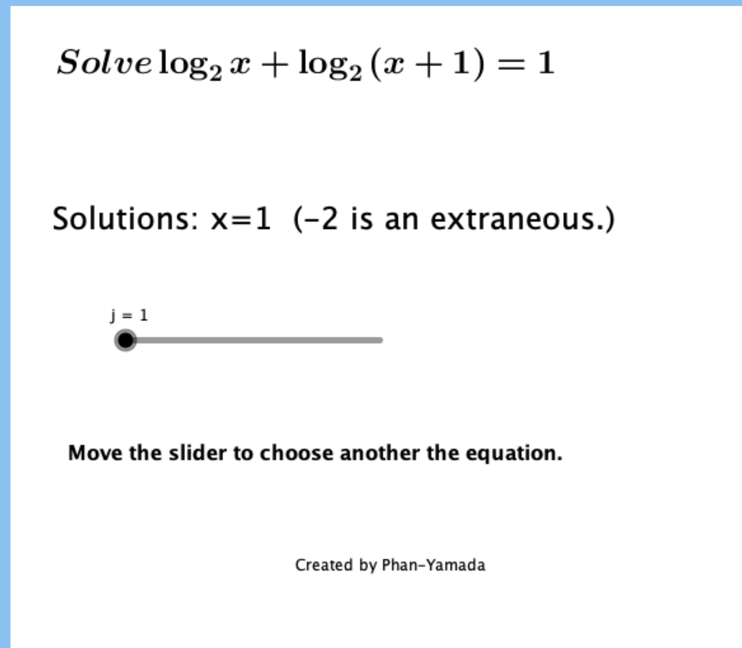
NICE EQUATIONS AND APPS TO TEACH MATH ONLINE



AMATYC Webinar June 19, 2020

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SYSTEM OF LINEAR EQUATIONS

For any system of linear equations

$$\begin{cases} ax + by + cz = r \\ dx + ey + fz = s \\ gx + hy + jz = t \end{cases}$$

Using the set of integers (x, y, z) with given coefficients, calculate r, s, t.

Then we can have an integer solution (x, y, z).

System of Linear Equation Bank

RATIONAL EQUATIONS

$\frac{1}{x} + \frac{1}{x-a} = \frac{1}{b}$ ➔ $x^2 - (a + 2b)x + ab = 0$

$x = \frac{2b+a\pm D}{2}$, where $D = \sqrt{(2b+a)^2 - 4ab} = \sqrt{a^2 + 4b^2}$

For any integers ***a*** and ***b*** , **x** is an integer if and only if **a^2 + $4b^2$** is a perfect square number, or (a, 2b, D) a Pythagorean triple.

So let ***a = 2n + 1*** and ***b = n(n + 1)***, then D = ***$2n^2 + 2n + 1$*** is an integer . Thus **2b + a ± D** is an even number since ***a*** and ***D*** are the same parity.

Therefore, $\frac{1}{x} + \frac{1}{x-a} = \frac{1}{b}$ has integer solutions ***$x=2n^2+4n+1$*** and $x = n$, where ***a = 2n + 1*** and ***b = n(n + 1)***, for some integer ***n.***

Rational Equation Bank

RADICAL EQUATIONS

Form 1: $\sqrt{x-a} = x - b$ → $x^2 - (2b+1)x + a + b^2 = 0$

Solution: $x = \frac{2b+1 \pm D}{2}$ where D = $\sqrt{(2b-1)^2 - 4a - 4b^2}$

Or D = $\sqrt{1-4(a-b)} \rightarrow D^2 = 1 - 4(a-b)$

If (***1 - 4(a - b)***) is a perfect square, then **D** is an odd number.

For any integer ***b*** and ***n*,** let ***D* = *2n* - *1*** and **a** = $\frac{\mathbf{1} - \boldsymbol{D}^{\mathbf{2}}}{\mathbf{4}} + \boldsymbol{b}$

then $\sqrt{\boldsymbol{x}-\boldsymbol{a}} = \boldsymbol{x} - \boldsymbol{b}$ has integer solutions: **x = b + n** and ***x = b - n+1***.
Note that if ***x*** < ***b*** , then x is an extraneous.

RADICAL EQUATIONS

Form 2: $\sqrt{cx - a} = x - b$ → $x^2 - (2b+c)x + a + b^2 = 0$

Solution: $x = \frac{2b+c \pm D}{2}$ where D $= \sqrt{(2b+c)^2 - 4a - 4b^2}$

Or D $= \sqrt{c^2 - 4a + 4bc}$. Again **c** and **D** are the same parity.

So if $\boldsymbol{D = c - 2n}$ and **a** $= \frac{\mathbf{4bc + c^2 - D^2}}{\mathbf{4}}$ = $\boldsymbol{bc + cn - n^2}$, then $\sqrt{cx - a} = x - b$ has integer solutions: $\boldsymbol{x = b + n}$ and $\boldsymbol{x = b + c + n}$.

Again, if **x < b** , then ***x*** is an extraneous.

RADICAL EQUATIONS

In summary, for any integers **b, n** and **c**.

If $\boldsymbol{a} = \mathbf{bc + cn - n^2}$, then

$$\sqrt{cx - a} = x - b \implies \mathbf{x = b + n} \text{ and } \boldsymbol{x} = \mathbf{b + c + n}$$

In general, $\sqrt{cx - a} = ex - b$ has integer solutions $x = p + q$ and $x = p$ for $b = pe$, $a = pqe^2$ and $c = qe^2$

Note that if $x < b$, then x is an extraneous.

Radical Equation Bank

LOGARITHMIC EQUATIONS

Form 1: $\mathbf{log_c(x - a) + log_c(x - b) = 1}$

$$x^2 - (a + b)x + ab - c = 0$$

$x = \frac{a+b\pm D}{2}$ where $D = \sqrt{(a + b)^2 - 4(ab - c)} = \sqrt{(a - b)^2 + 4c}$

Use the Pythagorean triple formula,

$\boldsymbol{a - b} = (2n + 1)$ *and* $4\boldsymbol{c} = \boldsymbol{4n^2(n+1)^2}$,

or $\boldsymbol{b = a - 2n - 1}$, $\boldsymbol{c} = \boldsymbol{n^2(n+1)^2}$ for any integer $\mathbf{n > 0}$,

then $D = \boldsymbol{2n^2 + 2n + 1}$.

Thus, **x** has two integer solutions $x = \boldsymbol{a + n^2}$ and $x = \boldsymbol{a - (n+1)^2}$.

Since $\boldsymbol{x = a + n^2 > a > b}$, it is always a solution.

However, $\boldsymbol{x = a - (n+1)^2}$ is an extraneous.

LOGARITHMIC EQUATIONS

<u>Form 2</u>: $log_c(x-a) + log_c(x-b) = 2$

$$x^2 - (a+b)x + ab - c^2 = 0$$

$x = \frac{a+b\pm D}{2}$ where D $= \sqrt{(a+b)^2 - 4(ab - c^2)}$

$D^2 = (a+b)^2 - 4\,(ab - c^2) = (a-b)^2 + 4c^2$. It is a Pythagorean triple!

So let (***a* – *b*) = *2n +1,*** or ***a = b + 2n + 1*** and ***c*** = ***n(n+1),*** then ***D = 2n² + 2n + 1*** for any integer **n** > **0.**

Thus $log_c(x-a) + log_c(x-b) = 2$ has integer solutions

$\mathbf{x = \frac{a+b+D}{2}}$ **= b + (n + 1)²** and

$\mathbf{x = \frac{a+b-D}{2}}$ **= b – n²** is an extraneous

Logarithmic Equation Bank

INTEGER TRIANGLES

Given a triangle with the integer lengths, find an angle with measurement in integer degrees.

The measure of one angle is either 60°, 120 ° or 90°

Use the Law of Cosine to calculate on Excel.

Integer **Triangles**

EIGENVALUES – 2X2 MATRICES

A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for ***a, b, c, d*** are integers.

Characteristics equation $\boldsymbol{\lambda^2 - (a+d)\,\lambda + ad - cb = 0}$

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4ad + 4bc}}{2}$$

$$= \frac{a + d \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

$$= \frac{a+d \pm D}{2}, \text{ for D} = \sqrt{(a-d)^2 + 4bc}$$

EIGENVALUES

$$D = \sqrt{(a-d)^2 + 4bc}$$

Again, (a-d, 4bc, D) is a Pythagorean triple. Thus, we can set

a – d =2n+1 or ***a= d + 2n + 1*** and 4***bc*** = ***4n²(n+1)²*** , or ***b =n², c =(n + 1)²***

So if ***d = a – 2n – 1, b = n², c*** = ***(n+1)²*** for any integer ***n***, then A has two integer eigenvalues of the form ***a + n²*** and ***a – (n+1)²***.

<u>Theorem :</u> ***For any integers a and n, the matrix*** A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Or $A = \begin{pmatrix} a & n^2 \\ (n+1)^2 & a-(2n+1) \end{pmatrix}$ ***has integer eigenvalues***

a + n² and a – (n+1)².

If a = - n² or a = (n+1)², then A has a zero eigenvalue.

EIGENVALUES -3X3 MATRICES

D = $\begin{pmatrix} s & t & r \\ p & a & b \\ q & c & d \end{pmatrix}$ for ***all entries*** are integers.

<u>Form 1</u>: ***p = q = 0***

Then the 3x3 matrix above becomes B = $\begin{pmatrix} s & t & r \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$ for

all entries are integers.

Its characteristics equation is ***(s - λ) (λ²*** ***– (a+d)λ + ad – cb)*** = 0.

Thus, the eigenvalues are ***s*** and ***a + 2n²*** and ***a – 2(n+1)²***

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} t+r \\ a+b-s \\ a+b-s \end{pmatrix}$, $\begin{pmatrix} rc-tb \\ b(s-a+c) \\ -c(s-a+c) \end{pmatrix}$ are the corresponding

eigenvectors of matrix B.

EIGENVALUES– 3X3 MATRICES

D = $\begin{pmatrix} s & t & r \\ p & a & b \\ q & c & d \end{pmatrix}$ for ***all entries*** are integers.

<u>**Form 2**</u>: ***t* = *b* = *q* = *c* = 0**

Then the 3x3 matrix above becomes C = $\begin{pmatrix} s & 0 & t \\ p & a & 0 \\ 0 & 0 & d \end{pmatrix}$ for ***all entries*** are integers.

Its characteristics equation is ***(s - λ)(a - λ)(d – λ)*** = 0.

Thus the eigenvalues are ***s, a,*** and ***d***.

$So\ one\ set\ of\ integer\ eigenvectors\ \ of\ matrix\ C\ could\ be$

$$\begin{pmatrix} s-a \\ p \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} t(d-a) \\ pt \\ (d-a)(d-s) \end{pmatrix}$$

<u>**Integer Eigenvalues**</u>

Integer Solutions

https://phan-yamada.weebly.com/

Click on **Teaching**

Select: **Math for Elementary and Middle School Teachers**

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Phan-Yamada Statistics Website:
https://sites.google.com/site/phanyamada/Home/teaching/statistics

https://www.geogebra.org/classic/khm2paqd

For any questions, please email me :
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Other teaching apps are posted on
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For any questions, please email me :
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