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AUTOMATED UPPER AND LOWER BOUND SOLUTIONS FOR THE ULTIMATE FLEXURAL CAPACITY OF CONCRETE SLABS

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ABSTRACT

This paper presents the accumulated output of over 30 years research that has resulted in the development of an automated method for predicting the ultimate flexural capacity of reinforced concrete slabs.

Initially, an automated yield-line analysis technique was developed. This was widely adopted in the UK for assessing the ultimate capacity of concrete bridges as part of a major national bridge assessment programme, however, as an "upper bound" or "unsafe" method, there remained some uncertainty over the validity of the library of failure mechanisms adopted and hence the reliability of the predicted capacity of the slabs. Next, a unique lower bound technique was developed which introduced the concept of "yield-line indicators" to generate a lower bound or "safe" prediction of the flexural capacity of a slab. The output of this program was used to manually generate a failure mechanism topology, which could then be used as an input to an improved upper bound computer program, resulting in both lower and upper bound predictions of the flexural capacity of the slab. More recently, an algorithm has been developed to automatically convert the lower bound yield-line indicator patterns into an upper bound yield-line mechanism, resulting in an automated methodology for generating both upper and lower bound predictions of the flexural capacity of reinforced concrete slabs.

An extensive validation exercise was undertaken to compare the predictions obtained using these upper and lower bound solutions for a number of case studies, mainly of bridges. The results showed that these independent analysis methods provided closely correlating predictions. This provided reassurance that the many upper bound assessments of bridges undertaken using only the original upper bound program provided reliable estimates of flexural capacity.

BACKGROUND

There has been a long history of research into plastic methods of analysis at Cambridge University starting with the work of Baker, Heyman and Horne during and after the Second World War. In 1988, a new programme of research aimed at developing a generalised method for predicting the ultimate flexural strength of reinforced concrete slab bridges was commenced, sponsored by the Transport Research Laboratory. At that time, the UK had embarked on a major bridge assessment programme to check that the existing stock of road bridges would be able to carry safely an increase in the legal lorry load limit that was to be introduced in 1999 to align with European Union standards. Although the planned increase from the then 38 tonne gross vehicle weight limit to 40 tonnes was in itself relatively small, the real issue was that the authorities did not actually know what the safe capacity or condition of much of their bridge stock was so this programme provided an opportunity to assess and, where necessary, strengthen or replace those bridges found to be inadequate. In the following years many tens of thousands of bridges were assessed, and thousands of these were deemed to be inadequate. As a result, hundreds of millions of pounds were spent on strengthening and replacing bridges throughout the UK. Even today, there remains a backlog of bridges that are still awaiting action.

However it was recognised from the very start of this programme that the elastic analysis methods commonly used in professional practice at the time were likely to be highly conservative in many instances and that the use of alternative analysis methods might result in a more realistic evaluation of the actual capacity of many of the bridges that were deemed to have "failed" the assessment process. One alternative was to employ non-linear finite element methods which were evolving rapidly and were quickly becoming the favoured analytical tool of researchers for predicting the behaviour of structures. However these tools were still predominantly used in the Universities or for highly specialised projects and were not widely adopted in industry. This was in part due to the high cost of the software but also due to the complexity and high level of expertise needed to apply and interpret these programmes.

An alternative approach, which had been widely used by researchers when studying the ultimate capacity of reinforced concrete slabs, was to employ yield-line analysis as this was found to be one of the most reliable and accurate analysis tools available. The major limitation of this method was that it relied on extensive and rather tedious hand calculations which evaluated a limited number of standard mechanisms which could only be applied to quite simple geometries, load-cases and reinforcement configurations. As a result, the objective set for the Cambridge research programme was to develop an automated, yield-line analysis computer programme for assessing the flexural strength of reinforced concrete slabs.

It is now thirty years since this research programme started and the remainder of this paper outlines the progress that has been made and the key steps in the evolution towards this objective.

A GENERALISED METHOD FOR YIELD-LINE ANALYSIS (COBRAS)

The initial conceptual breakthrough that allowed a generalised yield-line solution scheme to be computerised was the realisation that the yield-line problem could be reduced to what is fundamentally a problem of geometry. Although this might appear trivial to today's generation of engineers, it must be recognised that no general solution scheme has previously been developed. Using the then newly evolving developments in *computer graphics* and *solid modelling* theory, an analysis technique was developed which created a three-dimensional 'picture' of the bridge.

Perhaps the most significant feature of this modelling technique was its ability to analyse rigorously realistic configurations of loading, bridge geometry, support fixity and failure mechanisms without the need to derive mathematical expressions describing the inter-relationship between these parameters. Multi-layered, banded and curtailed reinforcement layers could also be included. It was even possible to make some provision for the effects of steel corrosion and concrete deterioration.

The solution scheme revolved around six tasks.

<u>Modelling the bridge and its structural components</u>

The fundamental parameters governing the collapse behaviour of a concrete bridge are the geometry in plan, the support fixity, the cross-sectional dimensions, the concrete strength and density, the details of the various layers of steel reinforcement and the applied loading. Each of these features are separately represented by polygonal shapes which are then merged or "intersected" together using computer graphics solid modelling techniques to form a single "bridge structure model" which represents the entire bridge and incorporates all the required analysis parameters of material components and geometry. For example, reinforcing steel can be defined by a polygon defining the outline plan of a layer of bars and also properties such as area of steel (per metre width), yield strength, effective depth, and orientation in plan.

The process of combining all the components together uses principles from set-theory, and the actual merging of component parts is performed using a generalised 3D solid modelling package, specifically written for this purpose.

<u>Building the applied load models</u>

Complex loading combinations allowing, for example, for lane loads, vehicle or individual wheel loads or line (knife-edge) loads are also represented by polygonal regions to which a given load intensity is applied. Since the magnitude and position of applied live loading is independent of the structural components of the bridge, the various load-cases to be assessed are "assembled" in the computer in the specified location on top of the structure model of the bridge deck. However they are not combined with the bridge model. In this way a separate, independent graphical representation of each load-case is stored in the computer enabling complex load combinations to be evaluated.

<u>Modelling the yield-line failure mechanisms</u>

The generalised analysis method generates 3D *polyhedral failure models*, which are 3D representations or "pictures" of each of the yield-line failure mechanisms chosen for analysis. These solid failure models provide all the required geometric information needed for the virtual work calculation used in the yield-line analysis method. One of the major strengths of this approach is that the failure modes are described totally independently of the load models and the bridge structure model, depending only on the shape of the bridge perimeter taken from the boundary representing the plan area of the bridge deck. By incorporating an extensive library of pre-defined yield-line mechanism topologies within the program, the user can easily choose an appropriate selection of collapse modes for assessment. This library includes a selection of some of the most commonly reported failure modes for bridge slabs and also some complex fan mechanisms.

<u>Creating a solid bridge model</u>

By merging the three models representing the *structure*, the *loading* and the *failure mechanism*, a single 3D solid model or "picture" of the entire bridge in its collapsed state is produced and stored in a data structure within the computer. This merging of the three component models to form a *solid bridge model* is accomplished using Boolean algorithms for graphical elements developed specifically for this purpose. The new solid bridge model contains all the information needed to perform a yield-line analysis, including the material components, dimensions, loading, failure mechanism topology as well as the location and length of all the yield lines, the details of abutment fixity at each of the boundaries and the relative rotations between adjacent rigid plate elements of the failure mechanism.

Optimising the failure mode geometry

The optimum or governing failure mode geometry is derived using a rapid "step-like" iteration of each of the selected failure mode topologies. This approach avoids the need to derive explicit equations for work done or energy dissipated or undertake an often-difficult partial differentiation calculation to obtain an estimate of the critical failure mode geometry. With a computer, a large number of iterations can be examined quickly, thus ensuring that the critical geometry for the particular mode is found to within the accuracy dictated by the selected iteration step size.

Calculating the ultimate strength and factor of safety

The final step is to calculate the load capacity of a bridge using the yield-line method in which a global *factor of safety (FOS)* is derived. Having applied a given assessment load to a postulated failure mechanism and derived a factor of safety, the parameters defining the failure mode geometry are varied to minimise the factor of safety and hence determine the load–capacity of the bridge.

The resulting computer programme, called COBRAS (for COncrete BRidge ASsessment), enabled structures that were hitherto impractical or impossible to assess by hand to be analysed automatically. With a modern portable computer a typical concrete bridge assessment can be performed in a couple of minutes.

LIMITATIONS OF THE COBRAS ANALYSIS METHOD

As an upper bound method of analysis, there is always a degree of uncertainty as to whether the critical mechanism topology and geometry has been identified. The current version of the COBRAS program relies upon selecting mechanisms from a pre-defined library of 27 failure mode topologies. The underlying data structures developed for modelling the structure, loading and failure mechanisms are, in principle, capable of being extended to any general polygonal shape however, the current commercialised version of the program is constrained to analyse four sided slab structures which must be square, rectangular or parallelogram in shape. In practise the vast majority of short span reinforced concrete bridges reviewed by the authors comply with these criteria. The method also relies upon the assumption that there is sufficient ductility for the full collapse mechanism to develop. It also does not evaluate the shear capacity and this must be considered separately.

LOWER BOUND (LB) AND UPPER BOUND (UB) PREDICTIONS OF ULTIMATE STRENGTH (PLASTISLAB)

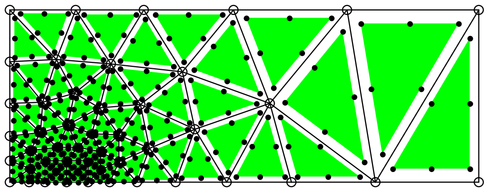
Having developed an automated, upper bound yield-line programme the next research objective was to develop a rigorous lower bound solution. This was one of the "holy grails" of concrete engineering and researchers had been attempting to find a robust, generalised lower bound solution for decades. In 2007, Andrew Jackson set about tackling this challenge for his PhD at Cambridge University.

To estimate a lower bound prediction of the collapse load of a slab it is necessary to find a moment field which is everywhere in equilibrium with the applied loads and nowhere exceeds the yield of the material. Any moment field complying with these criteria will give a safe estimate of the collapse load. Various lower bound methods, such as the Hillerborg Strip method, have been developed[3,4,5,6], but none has been widely applied to give accurate solutions to practical engineering structures. |To avoid overly conservative assessments, an optimisation technique must be used to find a combination of moments which result in the highest possible lower bound to the collapse load.

<u>Basic method</u>

The method developed by Jackson is derived from that used by Krabbenhoft and Damkilde[4] and implemented in the PlastiSlab program. As a plastic method of analysis, the slab is assumed to exhibit ductile, rigid-plastic behaviour. In addition, it only considers flexural behaviour; shear is not examined. For lightly reinforced concrete slabs these are reasonable assumptions but the validity of the results may be affected by brittle concrete crushing or reinforcement rupture in some situations. The new method also ignores membrane action, which may result in the true collapse load being significantly higher than predicted.

The slab is divided into a mesh of six-node triangular elements (Figure 1). The moment components m_x, m_y and m_{xy} are determined within each element by quadratically interpolating between the element's six nodes. A series of linear equilibrium equations on the nodal moments (Figure 1) ensure that the moments and applied load obey equilibrium everywhere[6].



Key		Equilibrium conditions
○	Mesh node	• Vertical force from adjacent elements and applied loads
___	Mesh edge	• Bending moment/length from adjacent elements and applied loads • Vertical force/length from adjacent elements and applied loads
	Mesh element	• Vertical pressure from element moment field and applied loads

Figure 1: A typical mesh and its associated equilibrium conditions

The yield condition is initially enforced at many 'control points', each with a set of nonlinear conditions on the moments to ensure that it is within the bi-conical Johansen yield surface[7] (Figure 4). Conic programming[4,8] finds the moment field which satisfies the equilibrium and yield conditions and has the highest collapse load. It should also be noted that the Johansen yield surface has been shown to be unconservative in some cases involving twisting moments[9].

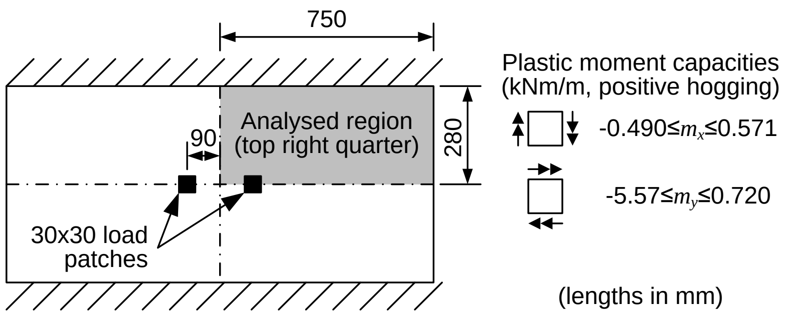


Figure 2: Collins's 1:10 scale reinforced concrete slab model

For example, Figure 2 shows a 1:10 scale model of a typical short-span concrete slab bridge which was tested in the laboratory at Cambridge University by Collins.[1,10] Collins measured an actual collapse load some three times greater than that predicted by elastic analysis. The slab was also analysed using COBRAS[1,10] and a lower bound modified Hillerborg Strip method[4].

Using a fine mesh, the conic programming stage of Jackson's new method initially predicted the moment field shown in Figure 3. As shown in Figure 5, the initial prediction of collapse load of 26.1kN lies between previous lower and upper bound analyses of the slab.

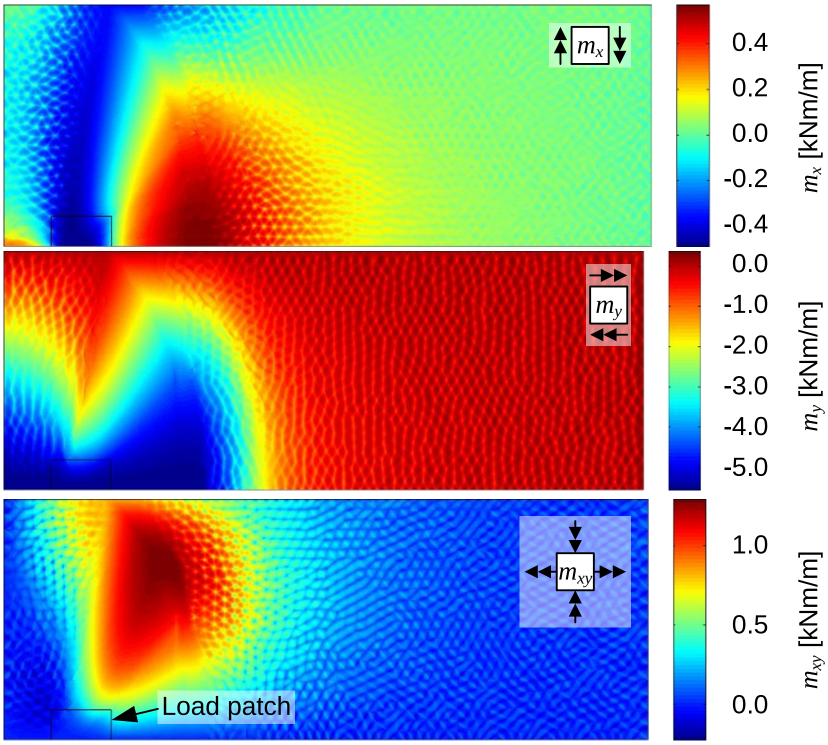


Figure 3: Moment fields for the top-right quarter of Collins's slab

<u>Modifying the results: R reduction</u>

Initially, this basic method only enforces the yield conditions at the control points, so the solution may exceed yield elsewhere in the slab. To obtain a rigorous lower bound to the collapse load, the applied loads and moments are then reduced until the moment field touches but does not cross the yield surface. At any point on the slab, a factor *R* describes the ratio by which the moments must be divided to just touch the yield surface. A numerical search is used to find the maximum *R* in the basic solution (e.g. point P in Figure 4). As long as the search has correctly found the maximum, reducing all the moments and applied loads from the basic method by this factor produces a rigorous lower bound solution. For the solution for Collins's slab shown in Figure 3, the maximum of *R* is 1.16, so 26.1/1.16 = 22.4kN is a new, lower bound to the collapse load.

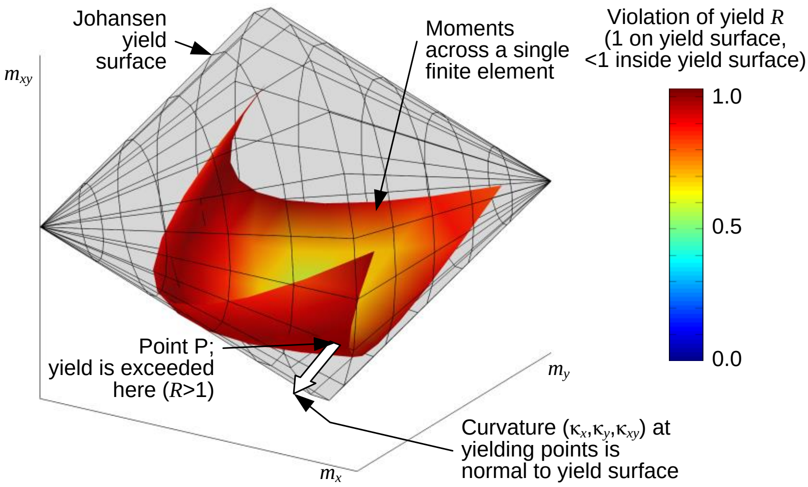


Figure 4: Using the yield surface to control yield and find curvature

Refining the results: control point addition

After *R* reduction the result is no longer optimal, and may have a collapse load considerably less than the exact plastic solution. This estimate can be improved by including more control points at positions where yield is most violated (local maxima of *R*, e.g. point P in Figure 4). After repeating the optimisation, the solution usually has a lower maximum violation of yield *R*, so the collapse load after *R* reduction is higher. This is continued until the maximum of *R* becomes acceptably small.

For Collins's slab, the predicted collapse load after control point addition and *R* reduction agrees with all previous analyses (Figure 5). It is 47% greater than the best previous lower bound analysis and within 2% of the best upper bound analysis. The actual collapse load of the experimental model was some 25% higher due to effects such as membrane action.

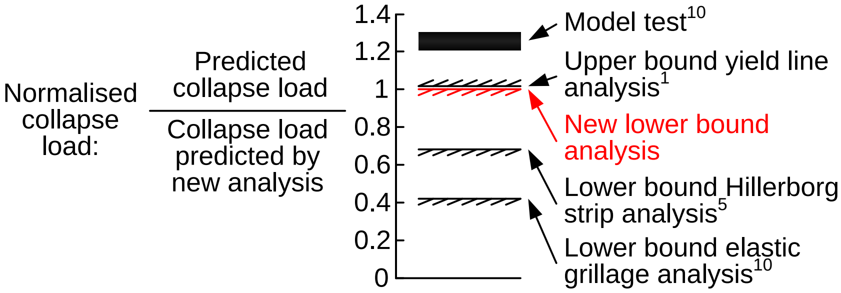


Figure 5: Predictions of the collapse load of Collins's Slab

Extending the LB method to generate a collapse mechanism for use in UB analysis

Moment fields are usually very difficult to check using 'engineering judgement' or simple hand calculations whereas the results obtained using upper bound collapse mechanism analysis are relatively easy to check. Jackson recognised that he could use his new lower bound analysis method, which he renamed as PlastiSlab LB, to generate a yield-line failure mechanism topology that could then be used as an input into a new upper bound yield-line analysis program. He also decided to develop his own version of an upper bound yield-line analysis program, which he called PlastiSlab UB. This modelled the slab structure as a triangulated mesh and was able to analyse more complex slab geometries than the COBRAS program. It was also not constrained to use mechanisms from the pre-defined library of mechanisms incorporated in the COBRAS program. Thus PlastiSlab UB would then derive a complementary upper bound prediction of collapse load which could be checked by hand. The gap between the upper and lower bound solutions would also give a measure of the maximum potential error in the estimated collapse load.

To generate an appropriate yield-line topology, consider the moment field in the final lower bound plastic solution. Within regions where yield is reached, the direction of curvature can be determined by the normality principle (Figure 4), and yield-lines can occur in the directions of principal curvature. It is therefore possible to plot a field of what are referred to as 'yield line indicators' from the lower bound solution. For example, Figure 6 shows that the yield line indicators from the lower bound solution for Collins's slab give a good prediction of the optimal yield-line pattern identified by COBRAS.

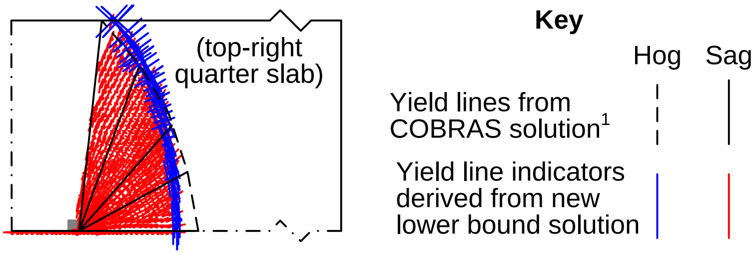
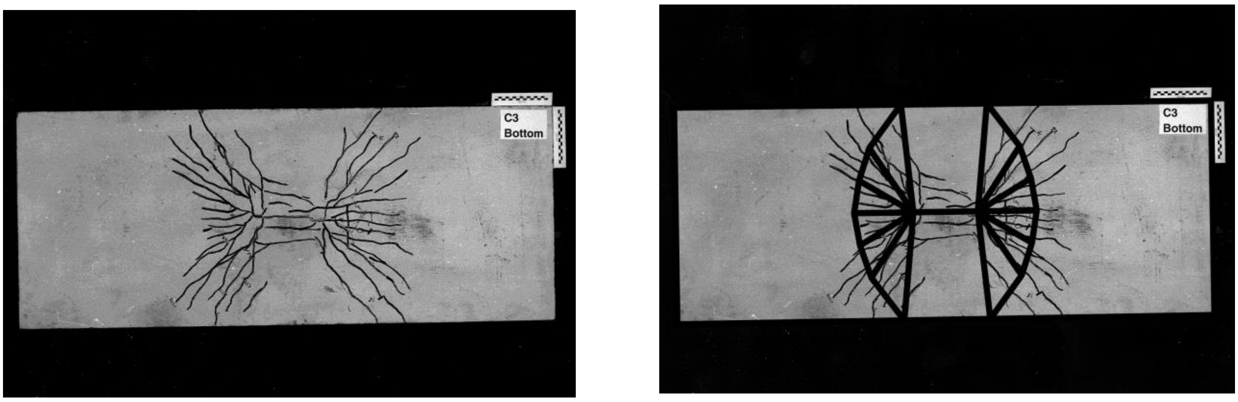


Figure 6: Comparison of yield-line collapse mechanism and the pattern of yield-line indicators for Collins's slab



(a) Collin's model slab – actual crack pattern

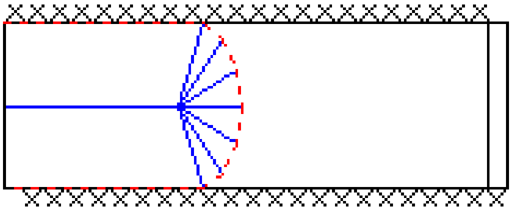
(b) Collin's model slab – actual crack pattern overlain with predicted yield-line mechanism

Figure 7. Photos of Collin's model slab at failure

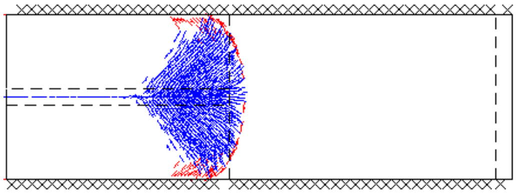
CALIBRATION OF THE COBRAS, PLASTISLAB LB AND PLASTISLAB UB PROGRAMS

An extensive calibration of the COBRAS and PlastiSlab computer programs was undertaken in 2016 by Cambridge University MEng student Grant Bellis. Bellis showed that for a number of example slabs, COBRAS and PlastiSlab LB & UB predicted failure loads that varied, on average, by only 3%, demonstrating a remarkable consistency. For comparison, Bellis also undertook analyses using both linear elastic and non-linear finite element analysis methods (NLFEA). He found that non-linear finite element analysis gave quite good comparisons with the PlastiSlab results but there was more inconsistency with NLFEA ranging from 8.3% lower to 14.4% higher than the PlastiSlab LB results. Bellis also showed that the failure load predicted by NLFEA was quite sensitive to the input parameters, such as material properties. This sensitivity is not an issue when using the COBRAS and PlastiSlab programs.

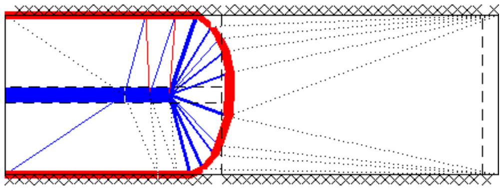
As an example, one of the bridge decks analysed by Bellis was the Blue House Bridge, a short-span bridge in Essex which carries traffic over the A128. The deck slab is rectangular and has fixed supports along both of the long abutments. It contains orthogonal layers of reinforcement in both the top and bottom of the slab. Figure 8 shows the results for the standard HA loadcase. COBRAS initial prediction of the FOS, using a coarse iteration mesh, was 1.210 however when reanalysed using a smaller, refined iteration step size, COBRAS predicted a FOS of 1.205. PlastiSlab LB gives a FOS of 1.166 whereas PlastiSlab UB predicts a FOS of 1.181. Using linear elastic FEA gives a FOS of 0.739 which would conventionally be interpreted to mean the slab was deemed to fail assessment. NLFEA predicts a FOS that is 7.2% higher than the PlastiSlab LB value.



(a) COBRAS - Critical Yield-line Mechanism



(b) PlastiSlab LB – Yield-line indicators



(c) PlastiSlab UB – Yield-line mechanism

Figure 8. Failure Mechanisms and Yield-line indicators

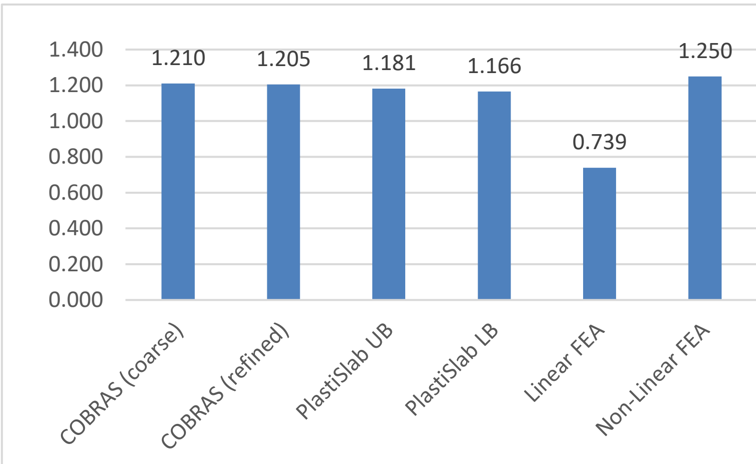


Figure 9. Factors of Safety for Blue House Bridge

LINKING JACKSON'S LOWER AND UPPER BOUND SOLUTIONS (DAYLI)

Jackson's program combined a lower bound equilibrium moment field (EMF) method with the upper bound yield-line method to produce an envelope around a slab's collapse load. The link between these two methods was provided by a manual editor in which the user was required to visually interpret the EMF solution to draw a likely yield line mechanism by hand. This was possible because the EMF results can be used to plot a pattern of elongated crosses, known as yield line indicators, on the surface of the slab. These use the normality principle to provide a visual indication of the locations and directions of yielding in the slab, and therefore where yield lines are likely to form.

For his final year MEng research project at Cambridge University, Andrew Smith was set the challenging task of developing an automated algorithm to replace the manual editor with an automated process, thus creating a self-contained, fully automated method that could be used to find upper and lower bounds on the ultimate collapse load of a reinforced concrete slab. In addition it was specified that the mechanisms produced should be simple, easy to visualise and straightforward to check. Smith went on to develop an innovative new algorithm which met these requirements. He named the algorithm DAYLI (Deterministic Algorithm for Yield Line Identification). The programme draws upon concepts from RANSAC (RANdom SAmple Consensus) – an algorithm often used in computer vision – and k-means clustering – a common data classification method. DAYLI is deterministic in the sense that it has no random steps; it will always give the same result from a given EMF solution. This is preferable because it ensures repeatability.

DAYLI works by fitting a number of straight lines and circles to carefully selected groups of yield line indicators in order to automatically generate possible yield line topologies that can be analysed using Jackson's PlastiSlab upper bound solver. The algorithm can easily be modified to incorporate other geometric models, such as ellipses for more complicated curved yield line indicator patterns or solid circular sectors for fans. An iterative averaging procedure is used to adjust lines and circles based on nearby yield line indicators to improve the fit. New lines are added until they are found to adequately fit most of the yield line indicators. Once a set of circles and lines has been identified, they are assembled into a yield line mechanism that mimics the pattern observed from the EMF solution. This requires truncating the lines and circles, approximating circular sections to a number of straight lines, and snapping together nearby points.

The functionality of DAYLI has been investigated with case studies which incorporate a range of boundary conditions, loadings, and reinforcement layouts. Six simple artificial slabs were devised to tune the algorithm, such as uniform squares simply supported on different numbers of sides and subjected to uniform pressure loads. Four more complicated real reinforced concrete slabs, including the Blue House Bridge deck, were used to show how DAYLI could be applied in real world assessments. These examples were also used to demonstrate some of the algorithm's limitations.

This is the first time that a solution has been found to generate upper bound yield line topologies automatically from lower bound solutions, thus producing bounds on the collapse load of an arbitrary slab. This represents a significant step towards a new fully automated program that could be used by practising engineers to determine the ultimate flexural collapse load of reinforced concrete slabs. There remains scope for further refinement of the EMF method, the development of more advanced geometric models for DAYLI, and investigating more efficient procedures for assembling yield line topologies.

CONCLUSIONS

This paper summarises the successful outcomes from 30 years of research aimed at developing improved methods for predicting the ultimate strength of reinforced concrete slabs. The first breakthrough was the development of a novel, automated, upper bound plastic yield-line collapse analysis program, called COBRAS, which enabled a wide range of slab geometries, reinforcement layouts and load cases to be analysed using a extensive library of possible failure mechanism geometries. Such analyses were hitherto impossible using analytical methods. One of the unique features of this approach was the fully automated procedure developed for combining all the structural components of a bridge (or slab) to produce a single solid model representing the bridge. A second key feature was the iteration algorithm invented to quickly optimise the topology of the failure modes selected by the user.

Subsequently, a rigorous lower bound computer program, called PlastiSlab, was developed which was complemented by a new, mesh based upper bound analysis program. Extensive calibration studies demonstrated that both the COBRAS upper bound program as well as the lower and upper bound PlastiSlab programs gave very closely correlated predictions for the ultimate flexural capacity of reinforced concrete slabs. The most recent contribution to this long term research programme was the development of an algorithm to automatically convert the yield-line indicators produced by the lower bound plastic collapse analysis program into a compatible and rational upper bound failure mechanism to act as a starting point for the new upper bound optimisation program. The development of this program provided the link between the lower and upper bound analysis tools hence allowing for fully automated analyses of the entire process resulting in complementary upper and lower bound predictions of flexural capacity. The closeness of the results obtained for the wide range of structures examined using these programs provided confidence in the validity of the independent methods of analysis that have been developed.

These developments offer engineers the tools needed to predict the flexural strength on ductile, reinforced concrete slabs.

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