Reflections on Good Calculus Questions from Students and Colleagues

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What is a “good” question?

• Based on a hunch/curiosity/coincidence
• Students get a taste of the nature of math
• Should have a reasonable degree of accessibility (e.g., Vygotsky’s ZPD)
• Investigation has the potential to...
  – Extend/generalize mathematical content
  – Lead to something new (or something already known but in a new light)
Counterexample

\[
\frac{\sin x}{n} = \frac{\sin x}{\kappa} = \text{six} = 6
\]

Again, why do the \(n\)'s cancel?
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The questions and potential “answers”

– Student-generated solutions
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THE QUESTIONS & SOME EXPLORATIONS
Question 1

Why is

\[ \left[ f(x) \cdot g(x) \right]' \neq f'(x) \cdot g'(x) ? \]

Spin the question as, “When, if ever, is \((f \cdot g)' = f' \cdot g'?\)"

\[
\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \frac{d}{dx}(g(x))
\]

\[
f(x)g(x) = \int f'(x)g'(x)\,dx
\]

Integrate by parts….

\[
u = f'(x) \quad dv = g'(x)\,dx
\]

\[
du = f''(x)\,dx \quad v = g(x)
\]

\[
f(x)g(x) = f'(x)g(x) - \int g(x)f''(x)\,dx \quad (1)
\]

Let \(f(x) = x^2\) so \(f''(x) = 2x\) and \(f''(x) = 2\). Now apply this to Eq (1):

\[
x^2g(x) = 2xg(x) - 2\int g(x)\,dx
\]
Student Work (continued)

Rearrange the above to get \((x - \frac{1}{2}x^2)g(x) = \int g(x) \, dx\). Differentiating both sides gives

\[
(x - \frac{1}{2}x^2)g'(x) + g(x)(1-x) = g(x).
\] (2)

Rearrange Eq (2) to \((2-x)\frac{dg}{dx} = 2g(x)\). Separation of variables and integration yield

\[
\int \frac{dg}{g} = \int \frac{2}{2-x} \, dx.
\] (3)

Solving Eq (3) for \(g\) gives \(g(x) = \frac{1}{(2-x)^2}\).

So we have the pair \(f(x) = x^2\) and \(g(x) = \frac{1}{(2-x)^2}\).
Can \((fg)' = f'g'\)?

\[
fg' + f'g = f'g' \\
fg' = f'g' - f'g = f'[g' - g]
\]

Try \(g = e^{ax}\), \(g' = ae^{ax}\).

\[
g' - g = ae^x - e^x = e^x(a - 1)
\]

\[
fae^{ax} = f'e^{ax}[a - 1] \\
f_a = f'[a - 1]
\]

\[
\frac{a}{a - 1} = \frac{f'}{f}
\]

\[
\frac{a}{a - 1} x = LN[f]
\]

\[
e^{\frac{a}{a-1}x} = f
\]

My ans: \(g = e^{ax}\) and \(f = e^{\frac{a}{a-1}x}\) so yeah, sometimes. Most likely other examples are possible. I checked too that both sides work out—wow!
Question 2

\[
\frac{d}{dr} (\pi r^2) = 2\pi r \quad \text{or} \quad \frac{d}{dr} (A_{\text{circle}}) = C_{\text{circle}}.
\]

Is this a neat coincidence or something deeper?

Perrin & Quinn (2008); Zazkis, Sinitsky, & Leikin (2013)
Analytical Approach

\[ A'(r) = \lim_{\Delta r \to 0} \frac{A(r + \Delta r) - A(r)}{\Delta r} \]

\[ = \lim_{\Delta r \to 0} \frac{\pi (r + \Delta r)^2 - \pi r^2}{\Delta r} \]

\[ = 2\pi r \]

= Circumference
Geometric Approach

Area of band \( \approx 2\pi r \cdot \Delta r \)

\( \Delta A \approx 2\pi r \cdot \Delta r \)

Radius increases by \( \Delta r \)

\[
\frac{\Delta A}{\Delta r} \approx \frac{2\pi r \cdot \Delta r}{\Delta r} = 2\pi r
\]
Generalizing

\[ \frac{d}{dr} \left( \frac{4}{3} \pi r^3 \right) = 4\pi r^2 \]
Generalizing

1. Why do these results not extend to squares or cubes? Or rather, what needs to be done to obtain consistency (Zazkis, Sinitsky & Leikin, 2013)?

2. How about polygons?

3. Platonic solids?
What is the point of writing "$dx$" when integrating? I keep forgetting to write it down!
Estimate the total force exerted on a window fully submerged under water.

Pressure varies with height so consider horizontal strips.

Force = (Pressure)(Area)

\[ \text{Force} \approx \sum_{k=1}^{n} P_k \Delta A \quad \Rightarrow \quad \text{Force} = \int_{\text{bottom strip}} P \, dA \quad \int_{\text{top strip}} \]

Jones (2013)
Student Work

Traditional:

\[
\int \tan^5 x \sec^2 x \, dx = \int u^5 \, du = \frac{u^6}{6} + C = \frac{\tan^6 x}{6} + C
\]

Better?

\[
\int \tan^5 x \sec^2 x \, dx = \int \tan^5 x \, d(\tan x) = \frac{\tan^6 x}{6} + C
\]
The symbols $dx$, $d(\tan x)$, $dr$, or $d($anything$)$ are more than placeholders telling us the “variable of interest.” Differentials have a tangible meaning through the operation of multiplication.
Question 4

When does

$$\int \sqrt{f(x)} \, dx = \sqrt{\int f(x) \, dx}$$
Students were asked to evaluate \( \int \sqrt{\tan x} \, dx \) (semester project).

1. Can I do \( \int \tan x \, dx \) instead?

2. Is \( \int \sqrt{\tan x} \, dx \) related in any way to \( \int \sqrt{\tan x} \, dx \)?
Direct attempt

When does \( \int \sqrt{f(x)} \, dx = \sqrt{\int f(x) \, dx} \) ?

We employed the standard notations \( F(x) = \int f(x) \, dx \) and \( F'(x) = f(x) \) and ignored all constants of integration for simplicity. Differentiating both sides of \( \int \sqrt{f(x)} \, dx = \sqrt{\int f(x) \, dx} \) followed by squaring leads to the (formidable) nonlinear differential equation \( F' = \frac{1}{4F}(F')^2 \). A classmate detected rather easily that \( F(x) = e^{4x} \) satisfied the differential equation. When we backpedaled and checked \( f(x) = 4e^{4x} \) in the original statement \( \int \sqrt{f(x)} \, dx = \sqrt{\int f(x) \, dx} \), we were pleasantly surprised.
An indirect attempt

The second line of thought involved using Hölder’s inequality
\[ \left| \int_a^b f(x)g(x)\,dx \right| \leq \sqrt{\int_a^b |f(x)|^2\,dx} \cdot \sqrt{\int_a^b |g(x)|^2\,dx} \] (a variation of this inequality appeared near the end of the exercise set as a theoretical exercise). We decided to put \( f(x) = \sqrt{\tan x} \) and \( g(x) = 1 \) and the above inequality reduced to
\[ \left| \int_a^b \sqrt{\tan x}\,dx \right| \leq \sqrt{b-a} \cdot \sqrt{\int_a^b \tan x\,dx} . \]

Students then reasoned it would be quite easy to produce a case where \( b-a = 1 \). For example, the above reduces to
\[ \left| \int_0^1 \sqrt{\tan x}\,dx \right| \leq \sqrt{\int_0^1 \tan x\,dx} . \]

Moreover, since \( \int_0^1 \sqrt{\tan x}\,dx > 0 \), we could say \( \int_0^1 \sqrt{\tan x}\,dx \leq \sqrt{\int_0^1 \tan x\,dx} . \)
Question 5

While using the Second Derivative Test, we found that all assumptions were met and \( f''(0) = 2 > 0 \). Hence, we concluded there was a relative minimum at \((0, f(0))\).

What is the meaning of the number 2?
Concavity vs. Curvature

What’s the difference?
"We can make another observation related to the degree of concavity (also called curvature). A large value of $|f''(a)|$ (large curvature) means ... the slope of the curve changes rapidly and the graph of $f$ separates quickly from the tangent line."

(Briggs, Cochran, Gillett, 2015)
"...absolute errors in linear approximation are larger when \(|f''(a)|\) is large."

Mostly true, but not always...
A "Turning Angle" $\theta$

$$\tan \theta = \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$
What is \( \frac{d\theta}{dt} \)?

\[
\tan \theta = \frac{dy}{dx} = \frac{y'(t)}{x'(t)}
\]

\[
\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{x'y'' - x''y'}{(x')^2}
\]

\[
\frac{d\theta}{dt} = \frac{x'y'' - x''y'}{(x')^2 + (y')^2}
\]

Implicit Differentiation + Quotient Rule

Solve for \( \frac{d\theta}{dt} \) and use

\[
\sec^2 \theta = 1 + \tan^2 \theta
\]

\[
\frac{d\theta}{dt} = \frac{f''(t)}{1 + (f'(t))^2}
\]
\[ \frac{d\theta}{dt} = \frac{f''(t)}{1 + (f'(t))^2} \]

At \( t = 0 \):
\[ f' = 0, f'' = 2 \]
\[ \Rightarrow \frac{d\theta}{dt} = 2 \text{ radians/unit}! \]

At \( t = 2 \):
\[ f' = 4, f'' = 2 \]
\[ \Rightarrow \frac{d\theta}{dt} = \frac{2}{17} \text{ radians/unit}! \]
Can we find the volumes of solids of revolution where the axis of revolution is a skew line such as $y = x$?
Idea

$y = f(x)$

$y = mx$

$P(a, f(a))$

$(x, f(x))$

$Q(b, f(b))$

$r(x)$

$a$

$x$

$b$
Challenge for the Student:
Find a "change of variables" from $\Delta x$ to $\Delta u$. 
Area and Volume Elements

Area Element:
\[ \Delta A = r \Delta u \]
\[ = \frac{(f(x) - mx)(1 + mf'(x))}{1 + m^2} \Delta x \]

Volume Element:
\[ \Delta V = \pi r^2 \Delta u \]
\[ = \pi \frac{(f(x) - mx)^2(1 + mf'(x))}{(1 + m^2)^{3/2}} \Delta x \]
Question 7

Is there an "integration by parts" for quotients?

Deveau & Hennigar (2012); Switkes (2005)
Why yes!

Begin with \( \left( \frac{u}{v} \right)' = \frac{vu' - uv'}{v^2} \). Integration wrt \( x \) gives \( \frac{u}{v} = \int \frac{vu'}{v^2} \, dx - \int \frac{uv'}{v^2} \, dx \) or

\[
\frac{u}{v} = \int \frac{vdu}{v^2} - \int \frac{udv}{v^2}.
\]

Simplify and rearrange to get

\[
\int \frac{udv}{v^2} = \int \frac{du}{v} - \frac{u}{v}.
\]
Features

\[ \int \frac{u \, dv}{v^2} = \int \frac{du}{v} - \frac{u}{v} \]

- Choose \( v \) (or \( v^2 \)), then find \( dv \), \( u \) must correspond to the remainder of the integrand.
- All we need on the right hand side is \( du \)!
Example (Standard)

\[
\int \frac{x^2}{(x^2+1)^2} \, dx = \int \frac{x^2+1}{(x^2+1)^2} \, dx - \int \frac{1}{(x^2+1)^2} \, dx
\]

\[= \int \frac{1}{x^2+1} \, dx - \int \frac{1}{(x^2+1)^2} \, dx \tag*{Let } x = \tan \theta\]

\[= \int \cos^2 \theta \, d\theta \]

\[= \int \frac{1 + \cos 2\theta}{2} \, d\theta \]

\[= \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta + C \]

\[= (\text{apply } \sin 2\theta = 2\sin \theta \cos \theta) \]

\[= \frac{1}{2} \tan^{-1} x + \frac{x}{2(x^2+1)} + C\]
Example (Quotient Rule IBP)

\[
\int \frac{u \, dv}{v^2} = \int \frac{du}{v} - \frac{u}{v}
\]

\[
\int \frac{x^2}{(x^2 + 1)^2} \, dx = \int \frac{1}{2} x \cdot 2x \, dx = \int \frac{1}{2} dx - \frac{1}{2} x \frac{x}{x^2 + 1} = \frac{1}{2} \tan^{-1} x - \frac{x}{2(x^2 + 1)} + C
\]

Integration by Parts

Discuss pros/cons of this formula with students!
Question 8

Which quantity is larger, 

\[ \int_{1}^{\infty} f(x) \, dx \text{ or } \sum_{n=1}^{\infty} f(n) \]
Hasty thought

\[ \int_{1}^{\infty} f(x) \, dx > \sum_{n=1}^{\infty} f(n) \]

Misconception: The “continuous” version is greater than the “discrete” version.
(i) \[ \int_{1}^{\infty} \frac{1}{x^2} \, dx = 1 \quad \text{whereas} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.65 \]

(ii) \[ \int_{1}^{\infty} \left(\frac{1}{2}\right)^x \, dx = \frac{1}{2 \ln 2} \approx 0.72 \quad \text{whereas} \quad \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \]

(iii) \[ \int_{1}^{\infty} xe^{-x^2} \, dx = \frac{1}{2e} \approx 0.18 \quad \text{while} \quad \sum_{n=1}^{\infty} \frac{n}{e^{n^2}} \approx 0.40 \]
More support for \[ \int_{1}^{\infty} f(x) \, dx < \sum_{n=1}^{\infty} f(n) \]

Re-examine the assumptions of the Integral Test!
Pedagogical Considerations

• The diagrams outline a proof of the Integral Test
• The proof re-ignites older material (Riemann Sums)
• Students need little convincing of the fact that if one converges, so does the other.

Challenge: Produce an example in which \( \int_{1}^{\infty} f(x) \, dx > \sum_{n=1}^{\infty} f(n) \). Explain how you arrived at your result.
Can an alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converge without the condition $a_{n+1} \leq a_n$?
\[
\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n) + 2}{4n^2}
\]

A list of \(a_1\) through \(a_{20}\):

\begin{align*}
0.7103677462, & \quad 0.1818310892, \quad 0.05947555578, \quad 0.01942496101, \quad 0.01041075725, \\
0.01194850349, & \quad 0.01355605407, \quad 0.01167718065, \quad 0.007444810139, \\
0.003639947223, & \quad 0.002066135937, \quad 0.002540672017, \quad 0.003580128753, \\
0.003814550198, & \quad 0.002944764267, \quad 0.001671969417, \quad 0.0008984450760, \\
0.0009637444088, & \quad 0.001488834632, \quad 0.001820590782
\end{align*}

A plot of \(y = a_n\) for large \(n\):
\[
\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \cdots
\]

\[
a_n = \begin{cases} 
\frac{1}{2^{(n+1)/2}}, & n = 1, 3, 5, 7, \ldots \\
\frac{1}{3^{n/2}}, & n = 2, 4, 6, 8, \ldots
\end{cases}
\]
Student 3 Solution

\[ a_1 = \frac{1}{2} \]
\[ a_3 = \frac{1}{4} + \frac{1}{2} \left( \frac{1}{4} \right) = \frac{3}{8} \]
\[ a_5 = \frac{1}{8} + \frac{1}{2} \left( \frac{1}{8} \right) = \frac{3}{16} \]
\[ a_7 = \frac{1}{16} + \frac{1}{2} \left( \frac{1}{16} \right) = \frac{3}{32} \]
\[ a_9 = \frac{1}{32} + \frac{1}{2} \left( \frac{1}{32} \right) = \frac{3}{64} \]
\[ \vdots \]

\[ a_8 = \frac{1}{4} \]
\[ a_6 = \frac{1}{8} \]
\[ a_4 = \frac{1}{16} \]
\[ a_2 = \frac{1}{32} \]
\[ a_1 = \frac{1}{64} \]
\[ \vdots \]
\[
\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{3}{8} - \frac{1}{8} \right) + \left( \frac{3}{16} - \frac{1}{16} \right) + \left( \frac{3}{32} - \frac{1}{32} \right) + \left( \frac{3}{64} - \frac{1}{64} \right) + \cdots \\
= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\
= \frac{1}{4} + \sum_{n=2}^{\infty} \left( \frac{1}{2} \right)^n \\
< \infty
\]
Student 4 (Beginning Idea)
Unexplored Questions
Question 10

Is Calculus much the same when working in degrees instead of radians?
Question 11

Are there coordinate systems that make the "business" of Calculus easier?
Given \[
\begin{align*}
  x &= a(\theta - \sin \theta) \\
  y &= a(1 - \cos \theta)
\end{align*}
\]
(parametric form for a cycloid),
why is \[
\frac{d}{d\theta}(x) = y?
\]
Question 13 (asked Fall 2015)

Given a convergent series \( \sum_{n=1}^{\infty} a_n \), will \( \sum_{n=1}^{\infty} (-1)^n a_n \) always converge?
We know $\lim_{n \to \infty} \left( \frac{n + 1}{n} \right)^3 = 1$ but why is $\lim_{n \to \infty} \left( \frac{n + 1}{n} \right)^n \neq 1$? Can one apply the "standard techniques" from the former problem to solve the latter?
WHAT IS THE POINT?
• Students are inherently curious (Harel’s intellectual need)
• Questions often connect to other areas of Calculus
• “New” mathematics may be just around the corner
ADVICE
• Don’t ignore these questions!
• Look for opportunities to use them in instruction
  • Class projects
  • Supplemental work
  • Let a student “show off”

**Takeaway**
All subjects are inherently inquiry-based. New knowledge is developed by asking questions and attempting to answer them!
References


Thank You!

Questions?

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