Abstract

Given that we have observed some green emeralds, why should we think that the next emerald is green rather than grue (where $x$ is grue iff $x$ is either green and observed before now, or blue and not yet observed)? This is the core question raised by Goodman’s New Riddle of Induction. In a Bayesian framework, this problem reduces to the choice of priors: Do we want a ‘green-friendly’ prior or a ‘grue-friendly’ prior?

I argue that there is good reason to prefer a green-friendly prior. Given that we are randomly sampling from a population of emeralds whose colors are determined independently of observation, our priors should satisfy certain symmetries. I proceed to show that these symmetries entail that evidence of observed green emeralds never confirms the prediction that the next emerald is grue, or the universal generalization that all emeralds are grue.

1 Introduction

Suppose I have observed a bunch of emeralds. All of these emeralds are green. Usually, we think that this evidence confirms that the next emerald I will observe is green, and perhaps also that all emeralds are green.

However, the emeralds I have observed are also all grue, where $x$ is grue iff $x$ is either green and observed before now, or blue and not yet observed (Goodman 1955, p. 74). Nonetheless, we normally think that the piece of
evidence that all emeralds observed so far are grue does not confirm that
the next emerald I will observe is grue (and hence blue), or that all emeralds
are grue. What explains this difference? This is the core question raised by
Goodman’s ‘New riddle of induction’.

In this paper, I want to propose an answer to this question, drawing on
the framework of Bayesian confirmation theory. The plan is thus as follows.
First, I sketch the basics of Bayesian Confirmation theory. Second, I in-
troduce a model of Goodman’s inference problem. Third, I argue that the
Bayesian can give a unified response to the different versions of Goodman’s
‘New riddle’ – given some substantive, but plausible assumptions about the
inference problem.

2 Bayesian Confirmation Theory

Bayesian confirmation theory consists of two basic ideas. First, we can
model the uncertainty of a rational agent by a prior probability distribution
$p$.

Second, we can define when evidence $E$ confirms hypothesis $H$:

$E$ confirms $H$ iff $p(H \mid E) > p(H),$

assuming that $p(E) > 0$.

3 A Bayesian Model

Before I explain my Bayesian response to Goodman’s ‘New riddle’, let me
first introduce a model of the problem. This model will help us to clarify
what the Bayesian needs to do in order to respond to the problem.

The set-up of Goodman’s ‘New riddle’ is as follows: We are observing
emeralds. Given evidence about the color of observed emeralds, we consider
the confirmation of different hypotheses about unobserved emeralds.

1For overviews, see Earman (1992), Urbach and Howson (1993), and Strevens (2017).
2Strictly speaking, we model the uncertainty of a rational agent as a probability space
$(\Omega, \mathcal{F}, p)$, where $\Omega$ is a nonempty set representing epistemic possibilities, $\mathcal{F}$ is a $\sigma$-algebra
of subsets of $\Omega$ and $p$ is a probability measure on $\mathcal{F}$. 
Let us model this inference problem as follows. Consider a countably infinite sequence of random variables \( X_1, X_2, X_3, \ldots \). \( X_1 \) is the color of the first emerald, \( X_2 \) is the color of the second emerald, and so on. For the sake of simplicity, let us assume that all emeralds are known to be either green or blue. Thus, we have

\[
X_i = \begin{cases} 
1 & \text{if the } i\text{-th emerald is green,} \\
0 & \text{if the } i\text{-th emerald is blue.}
\end{cases}
\]

(You might worry that we are begging the question by describing the problem in terms of green rather than grue. I will discuss this worry below.)

In this model, we can formalize all the essential ingredients of Goodman’s problem. Our evidence is that all \( k \) emeralds observed so far are green. Thus, we can define our evidence proposition \( E \) as follows: for all \( i \in \mathbb{N} \), if \( i \leq k \), then \( X_i = 1 \). Note that this is equivalent to saying that all emeralds observed so far are grue. This is because, for observed emeralds \((i \leq k)\), being green and being grue are equivalent, given the definition of grue: the \( i \)-th emerald is grue iff either it is green and \( i \leq k \), or it is blue and \( k < i \).

Now, we can study how our evidence affects the probability of various hypotheses. First, all-green says that all emeralds are green, so \( X_i = 1 \) for all \( i \in \mathbb{N} \). In contrast, all-grue says that all emeralds observed before (and including) the \( k \)-th emerald are green, and the rest is blue. So for all \( i \in \mathbb{N} \), we have:

\[
X_i = \begin{cases} 
1 & \text{if } i \leq k \\
0 & \text{otherwise.}
\end{cases}
\]

Second, the prediction next-green says that the next emerald is green: \( X_{k+1} = 1 \). In contrast, the prediction next-grue says that the next emerald is grue, and hence blue, so \( X_{k+1} = 0 \).

With these ingredients in place, we can ask what the Bayesian needs to do in order to solve Goodman’s problem. First, does our evidence \( E \) confirm all-grue? Note that, in our model, all-grue entails \( E \): if all emeralds observed before \( k \) are green and the rest is blue, it follows that all emeralds observed before \( k \) are green, which is exactly what \( E \) asserts. Now observe
the following:

**Fact 1.** If $H$ entails $E$ and $1 > p(H) > 0$ and $1 > p(E) > 0$, then $E$ confirms $H$.

This means that if we assign priors strictly between zero and one to both all-grue and $E$, it follows that $E$ confirms all-grue. Now I take it that any reasonable prior will assign some probability strictly between zero and one to the proposition that the first $k$ emeralds are green, and so to $E$. This severely limits the options for the Bayesian to respond to the first version of Goodman’s problem. The only way to deny that $E$ confirms all-grue is to argue that we should assign a prior of zero to all-grue.

Now we could just stipulate that the prior probability of all-grue is zero. Perhaps this is the best description of our inductive practice. However, just asserting as brute fact that all-grue has zero prior probability does not seem satisfying. We would like some explanation of why all-grue should be assign zero prior probability. Furthermore, even if we assign a prior probability of zero to all-grue, this does not yet answer the second version of Goodman’s problem: does evidence $E$ confirm next-grue? It might be that $p($all-grue$) = 0$, but $E$ still confirms next-grue. Thus, simply stipulating that the prior probability of all-grue is zero does not seem like a good response to Goodman’s problem.

Let us move on and ask: What does the Bayesian need to do in order to answer the second version of Goodman’s problem, viz. does $E$ confirm next-grue? Note that next-grue does not entail $E$. Hence, if we assign non-zero priors to both next-grue and $E$, nothing follows about whether $E$ confirms next-grue. Rather, it turns out that $E$ confirms next-grue just in case the covariance of $E$ and next-grue is positive: $Cov(E, \text{next-grue}) > 0$, where

\begin{equation}
Cov(E, \text{next-grue}) = \frac{\sum_{i=1}^{n} (E_i \times \text{next-grue}_i) - \bar{E} \times \bar{\text{next-grue}}}{\sqrt{\sum_{i=1}^{n} (E_i - \bar{E})^2} \sqrt{\sum_{i=1}^{n} (\text{next-grue}_i - \bar{\text{next-grue}})^2}}.
\end{equation}

$\text{Proof:}$ Suppose $H$ entails $E$, so $p(E \mid H) = 1$. Suppose further that $1 > p(H) > 0$ and $1 > p(E) > 0$. Now $E$ confirms $H$ iff $p(H \mid E) > p(H)$, and

\[ p(H \mid E) = \frac{p(H)p(E \mid H)}{p(E)} = \frac{p(H)}{p(E)}. \]

and by the assumption that $1 > p(H) > 0$ and $1 > p(E) > 0$, $\frac{p(H)}{p(E)} > p(H)$.

$^4$Responding to the problem by assigning a prior of one to all-grue is a non-starter.
\[ \text{Cov}(X,Y) = p(X \cap Y) - p(X)p(Y) \] (Sober 1994, p. 233).

I conclude that in order to solve the second version of Goodman’s problem, the Bayesian needs to give an argument for adopting a prior probability distribution on which the covariance of \( E \) and next-grue can never be positive.

### 4 The Symmetry Assumption

In this section, I introduce a constraint on priors which allows the Bayesian to respond to Goodman’s problem. Further, I argue that the constraint corresponds to plausible assumptions about the inference problem: We are randomly sampling from a population of emeralds whose colors are determined independently of our observation.

First, we need to introduce some terminology:

**Definition 1.** A (finite) sequence of random variables \( X_1, X_2, \ldots, X_n \) is exchangeable if the distribution of \( X_1, \ldots, X_n \) is the same as the distribution of \( X_{\pi(1)}, \ldots, X_{\pi(n)} \) for every permutation \( \pi \) on \( \{1, \ldots, n\} \) (Pitman 1993, p. 237).

To say that a sequence of random variables is exchangeable thus means that, fixing an assignment of values to the random variables, permutations of indices among the random variables do not change the probability of the assignment of values. Intuitively, this means that order doesn’t matter: The probability of a given assignment is determined solely by which values we get, not by the order in which we get them.

We can extend this definition to infinite sequences as follows:

**Definition 2.** A (infinite) sequence of random variables \( X_1, X_2, \ldots \) is exchangeable if every finite subsequence \( X_1, \ldots, X_n \) is exchangeable (Zabell 2011, p. 290).

---

\(^{5}\)Proof: \( E \) confirms \( H \) iff \( p(H \mid E) > p(H) \), and

\[
p(H \mid E) > p(H) \iff \frac{p(H \cap E)}{p(E)} > p(H) \iff p(H \cap E) > p(H)p(E) \iff \text{Cov}(E,H) > 0.
\]
Now I propose the following constraint on priors in Goodman’s inference problem. Recall that we modeled this problem as sequence of random variables $X_1, X_2, \ldots$, which indicate the color of the first observed emerald, the second observed emerald and so on. I claim that we should require:

**Symmetry.** The sequence $X_1, X_2, \ldots$ is exchangeable.

This principle states the following. Suppose that during our first $n$ observations, we observe $k$ green and $n-k$ blue emeralds. By **Symmetry**, it follows that all specific orderings of these emeralds are equally likely. (Note that there are $\binom{n}{k}$ such orderings.\(^6\)) In other words: The prior probability of observing a certain number of green and a certain number of blue emeralds is invariant under permutations of the order in which the emeralds are observed.

Now why should we assume **Symmetry**? My answer: We should assume **Symmetry** given that we are randomly sampling from a population of emeralds, whose colors are fixed independently of the order in which we observe them.

Let me explain. Suppose there is a population of emeralds whose colors (green or blue) are already determined. Now we randomly sample from this population. This means that each emerald has the same probability of being chosen. It follows that the color of the emeralds does not affect the order in which we observe them. Thus, we have **Symmetry**. Note that this fits well with the practice of statistics, where the symmetry of random variables is a standard assumption to model random sampling.

Note that **Symmetry** does not follow from the axioms of probability calculus. (Later, we will see concrete examples of priors which violate symmetry.) Rather, **Symmetry** embodies two substantive assumptions about the world:

1. The colors of emeralds are determined independently of the order in which we observe them,

2. We are randomly sampling.

---

\(^6\binom{n}{k}\), pronounced ‘$n$ choose $k$’, is defined as $\frac{n!}{k!(n-k)!}$.
Both of these assumptions might turn out to be false. However, my claim is that these assumptions are very plausible in Goodman’s inference problem. Further, my claim is that if we are willing to make these assumptions, we can solve Goodman’s problem. Still further, I would like to suggest that if we don’t have any reason to think that we are randomly sampling from a population of emeralds whose colors are determined independently of our sampling procedure, this means that we don’t have any good reason to prefer green-ish inductive inferences over their grue-ish counterparts. Last but not least, note that while these are substantive empirical assumptions, they are not the assumption that ‘green’ is somehow better or more natural than ‘grue’. They are assumptions about the world, not about the right way to ‘carve up’ the world. We will return to this point below.

After explaining and defending SYMMETRY, let me move on to explain how it solves Goodman’s problem. Recall that the first version of Goodman’s problem was the question: Is all-grue confirmed by $E$? As explained above, the only plausible way for the Bayesian to give a negative answer to this question is to argue that all-grue should be assigned a prior probability of zero. Now, perhaps somewhat surprisingly, we have the following:

Theorem 1. Any prior $p$ over the sequence $X_1, X_2, ...$ which satisfies SYMMETRY assigns zero prior probability to all-grue.

The proof is in the appendix. Thus, if we are willing to assume SYMMETRY, we are thereby committed to assign zero prior probability to all-grue. Therefore, evidence $E$ does not confirm all-grue, for the simple reason that hypotheses with zero prior probability cannot be confirmed by any evidence whatsoever.

Note that SYMMETRY does not force us to assign zero prior probability to all-green. It is compatible with SYMMETRY that all-green is assigned a non-zero prior. If we chose to do this, it follows that $E$ confirms all-green, but

---

7Note that Godfrey-Smith (2002) also emphasizes the importance of random sampling and the observation-independence of the properties under investigation for addressing the ‘New riddle’. However, Godfrey-Smith does not investigate how these ideas can be captured in a Bayesian framework.
not all-grue. Thus, Symmetry allows the Bayesian to give a good response to the first version of Goodman’s problem.

Let me move on to the second version of Goodman’s problem. This is the question: Is next-grue confirmed by $E$? As explained earlier, in order to respond to this question, the Bayesian needs to argue that the covariance of $E$ and next-grue can never be positive, so $\text{Cov}(E, \text{next-grue}) \leq 0$.

Luckily, it turns out that Symmetry helps with this problem as well. We have the following:

**Theorem 2.** Any prior $p$ which satisfies Symmetry has $\text{Cov}(X_i, X_j) \geq 0$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$ with $i \neq j$.

This means the following. If we accept Symmetry, it follows that the covariance of observed green emeralds and future green emeralds is greater than or equal to zero. Now the next emerald is grue iff it is not green. Therefore, the covariance of observed grue emeralds and future grue emeralds is always less than or equal to zero.

Let me sum up my argument. I argued that we should assume Symmetry: Given that we are randomly sampling from a population of emeralds with colors determined independently of our observation, we should think that any ordering of a given number of green and blue emeralds is equally likely. I further argued that given this assumption, the Bayesian can solve both versions of Goodman’s problem.

## 5 Grue-ish symmetry

I argued that we should be symmetric with respect to green and blue: Any ordering of a fixed number of green and blue emeralds is equally likely. As suggested earlier, you might suspect that this incorporates some sort of bias towards ‘green-ish’ descriptions over ‘grue-ish’ descriptions, and therefore begs the question against Goodman. You might worry that

[w]hat we seem to have here is simply some sort of ‘a priori’ prejudice against ‘grue’ hypotheses enshrined in Bayesian formalism. (Fitelson [2008, p. 631]
To make this worry more vivid, recall that we defined our sequence of random variables $X_1, X_2, \ldots$ in terms of ‘green’ and ‘blue’. In particular, we said that $X_1 = 1$ iff the $i$-th observed emerald is green. However, nothing stops us from defining an alternative sequence of random variables $Y_1, Y_2, \ldots$ in terms of ‘grue’ and ‘bleen’. (Where $x$ is bleen iff $x$ is either blue and observed or green and not yet observed.) In particular, let us define:

$$Y_i = \begin{cases} 
1 & \text{if the } i\text{-th emerald is grue}, \\
0 & \text{if the } i\text{-th emerald is bleen}, 
\end{cases}$$

which is equivalent to

$$Y_i = \begin{cases} 
1 & \text{if either } i \leq k \text{ and the } i\text{-th emerald is green or } k < i \text{ and the } i\text{-th emerald is blue}, \\
0 & \text{if either } i \leq k \text{ and the } i\text{-th emerald is blue or } k < i \text{ and the } i\text{-th emerald is green}. 
\end{cases}$$

Now here is the challenge: Why not be symmetric with respect to the sequence $Y_1, Y_2, \ldots$? That is, why not require:

**Symmetry*. The sequence $Y_1, Y_2, \ldots$ is exchangeable.

Observe that if we assume **Symmetry* instead of **Symmetry, we can mirror the arguments I have given earlier to show that all-green must be assigned a prior probability of zero and that the covariance between $E$ and next-green can never be positive. Thus, it looks like we have not escaped Goodman’s challenge at all. The defender of ‘grue’ can use mirrored versions of my arguments to ‘prove’ that grue-friendly priors are superior to green-friendly priors.

Here is my reply to this objection. The line of reasoning just sketched out shows that a ‘grue-friendly’ prior is perfectly consistent with the framework of Bayesian confirmation theory. This means that we cannot solve Goodman’s problem by only appealing to the axioms of probability theory, conditionalization and the Bayesian definition of confirmation. However, my argument for **Symmetry was not based on purely ‘formal’ considerations. Rather, I argued that we should adopt **Symmetry because it formalizes plausible substantive assumptions about our inference problem, viz. that
we are randomly sampling and that the colors of emeralds are determined independently of the order in which we observe them.

Once we take these substantive assumptions into consideration, the argument for green-friendly priors and grue-friendly priors is not longer symmetrical. Plausibly, emeralds are green or blue independently of the order in which we observe them. However, whether emeralds are grue or bleen is not independent of the order in which we observe them.

This point has been emphasized before by Jackson (1975), who argued that grue is not ‘counterfactually independent’ of observation. Let me explain this point. If an observed emerald is green, it still would have been green if we hadn’t observed it. However, if an observed emerald is grue, it would not have been grue if we hadn’t observed it. This is because an emerald is grue iff it is either observed and green or unobserved and blue. Suppose an emerald is grue and observed, and thus green. Had we not observed this emerald, it still would have been green. Therefore, had we not observed this emerald, it would have been green and unobserved, and so not grue.

The line of reasoning just sketched relies on substantive empirical assumptions about the world. However, these assumptions are not that green is ‘more natural’ than grue (Lewis 1983), or that it is somehow intrinsically ‘better’ to think and speak in a green-ish language rather than a grue-ish language. (Perhaps because such a language ‘carves at the joints’, mirroring the objective structure of the world, as argued by Sider (2011).) Rather, the assumption needed for our argument is much more pedestrian: green is counterfactually independent of observation, but grue is not.

References


