Modelling Social Enterprises*

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Abstract

This paper formally describes social enterprises taking into consideration their main aims. Based on the ‘earned income’ school of thought, the principal-agent theory and considering the nature of social enterprises’ activities we propose the objective functions of their owners (principals) and managers (agents). The maximisation problem of the social enterprise is defined as the weighted average of the utilities of two groups of stakeholders who have an influence on the degree of the realisation of the social mission and business orientations. We point out the direction which social enterprises should follow in order to obtain the highest value of their objective functions. The desired state for social enterprises should be one in which they reinvest all surpluses in the process of their mission realisation and their profit is slightly higher than the required level by the principal’s contract.

Keywords: Social enterprises, social activities, social mission, principal-agent theory, objective functions.

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1 Introduction

For many years social enterprises (SEs) have played a key role in society and business life. Apart from realising the social mission, they improve the global economy by reducing the number of social pathologies and diminishing inequalities. However, one can point out several differences between SEs and traditional businesses, in terms of the incentives to operate, the goals to achieve or the sources of financing. This distinction, which calls for establishing a completely novel theoretical approach, opens many new paths for the fundamental research.

There are three generic forms of organisations (Jegers, 2011b): governments, profit organisations (POs) and non-profit organisations (NPOs). The majority of researchers restrict their analyses to these kinds of institutions, while many enterprises are mixtures of these forms. One of the hybrid forms is the SE. According the European Research Network (EMES) definition of the SE (Defourny and Nyssens, 2006), it should satisfy the following criteria: it operates continuously, has a high autonomy, has a considerable level of risk and employs a minimum number of paid workers. Further, it should encapsulate a clear goal to advance the community, be launched by a group of citizens and allow only a limited distribution of profits. In practice, the SE is a ‘mishmash’ of legal forms and projects (Young, 2012), with a combination of goals, resources and globally institutional logics (Billis, 2010; Nyssens, 2006). That is exactly the reason why they are called hybrid organisations (Defourny and Pestoff, 2008; Low, 2006). An example of a hybrid form might be a microfinance organisation which faces a double challenge (Battilana and Dorado, 2010), commonly called a ‘double bottom line’ (Cornée and Szafarz, 2013): the mission’s realisation goes hand in hand with minimal profitability (Dart, 2004).

Currently, scholarly analyses of a fundamental framework regarding the activities of SEs are limited. With this paper, we aim to close this research gap by proposing a formal environment in which SEs are active. To do this, we propose a general model which describes the SE as a hybrid form. We look for a way to present the social mission and the activities of SEs, taking into consideration the financial return and the mission’s realisation. Our model describes an SE as an organisation which combines mission realisation with business activities. The SE receives donations and subsidies like an NPO and runs...
a profitable business like an PO. The maximisation problem of the SE is determined by a mixture of the objectives of two groups of stakeholders: owners and managers.

The structure of the rest of the paper is as follows. In the following sections, we present a review of the current literature, the theoretical model and our assumptions. Then, in section 4 we describe the model specifications, formulate our propositions and analyse the impact of parameters values. The paper ends with a concluding section.

2 Literature review

Due to the lack of theoretical references of SEs, this section relies on the NPOs’ literature on modelling NPOs. Moreover, conceptual studies about social entrepreneurship are presented to show which features of these entities are the most relevant and which of them should be taken into consideration when defining the theoretical framework of activities of SEs.

Malani et al. (2003) define a general model of NPOs and POs, which in fact is a profit maximisation model. Depending on the final results of the NPO and the PO, including all benefits from the NPO status, owners choose to institutionalise their organisation as an NPO or an PO. The authors conclude that if a firm is truly altruistic, it would not take the PO status. This research shows two attitudes which are opposed to themselves and enable us to analyse the hybrid enterprise.

The analysis presented by Wedig (1994) stresses differences in preferences of managers and donors. He constructs a dynamic corporate financial model which presents the relationship between donors and managers in NPOs. He splits the objective of the NPO into two groups of stakeholders with different goals. Managers strive to maximise their utility selecting the debt level and the dividend level, while donors determine the donation level, taking into consideration the NPO’s borrowing level and dividends level. This motivation is adopted in our paper to show differences between managers and owners.

Wedig et al. (1988) propose value maximisation models in both for-profit and non-profit hospitals. In both cases, the main feature is the maximisation of cash flow, but for different reasons. Non-profit hospitals try to achieve nonfinancial profits, such as a higher level of services, rather than profit maximisation, while for-profit hospitals paying
dividends to owners is crucial. In their empirical part (Wedig et al., 1988) they study US hospitals and find that the influence of changes in hospitals’ cost-base depends on decisions about the capital structure. Bowman (2002) broadens this approach by including universities and colleges, cultural and art organisations and human services agencies. He shows that previous empirical studies of the capital structure in non-profit hospitals are not representative. He calls for a better understanding of the internal situations of NPOs and external regulations. Both papers pinpoint differences between organisations which are active in the same domain but do not have the same mission. Our model enables us to implement these kinds of differences and make a significant distinction in the mission perception.

The paper from Weerawardena and Sullivan Mort (2006) points out key components with which the SE should struggle, but unfortunately it does not define in a formal way in which the organisation should deal with these aspects. They develop a multidimensional model of social entrepreneurship where they describe social value creation, taking into consideration innovativeness, reactiveness and risk management. They propose that social value creation is the outcome of social entrepreneurship. Their model is formulated in terms of the organisation’s sustainability, its social mission and its environment. We conclude that this approach boils down to dealing with a ‘triple bottom line’ which in our general model problem is reduced to ‘double bottom line’. However, the pillar of environment may be implemented in the proposed social mission function and then in the special case it is possible to analyse an SE which faces ‘triple bottom line’.

Zahra et al. (2009) define social entrepreneurship and propose ‘Total Wealth’ (TW) as a broader term which reflects economic wealth and social wealth. In the TW they include economic and social values which are reduced by economic, social and opportunity costs. This TW is adopted in our theoretical model and results in supporting social problems, realising social missions and running sufficiently profitable business.

Austin et al. (2006) model social entrepreneurship by applying the PCDO (People, Context, Deal, Opportunity) framework which was introduced for POs by Sahlman (1996). They stress the social value proposition (SVP) which is the result of the role of the social mission and the nature of the SE. Their model puts the SVP between people, opportunity and capital which/who are involved in the core of the problem of the social
entrepreneurship. Furthermore, they describe the environment where the SE runs its business. Authors focus on tax, regulatory, demographics, sociocultural, political and macroeconomy aspects. This frame gives us a clear idea about SEs and confirms the importance of the defining the social mission within each organisation separately.

According to the basic structure of an NPOs theory, the board of an NPO is the principal which internalises the social mission and has clear goals and objectives, while the manager is the agent (Jegers, 2011b). Features which distinguish them from managers in POs are their lower financial expectations and their higher altruistic motivations. In the literature overview by Jegers (2011b), however, it is underlined that NPOs can suffer from the same governance problems as POs. In some cases they can be even more extreme. The approach proposed by Steinberg (1986) and applied by Jegers (2011a) allows for a precise analysis of the principal-agent (board-manager) relationship and the balance of power between them. Steinberg (1986) schematically describes the NPO’s objective as a combination of the board’s goal (activity) and the manager’s objective (budget), weighted by objective function parameters $k$ and $(1 - k)$. In the borderline cases, the organisation maximises its activities ($k = 1$) or maximises its budget ($k = 0$). Jegers (2011a) used Steinberg’s approach to model an NPO utility function and presents the impact of agency problems (reflected by $(1 - k)$). These studies give us a strong motivation to define the compromise between our stakeholders. In our model, this approach is implemented but we propose different individual objectives of involved actors.

Van Puyvelde et al. (2012) analyse stakeholder theory and stewardship theory next to empirical research on the governance and management in NPOs. They identify actors in NPOs and describe external and internal relationships between a principal and an agent. They conclude that the internal agency problems should be avoided in NPOs. This paper confirms that there is a principal-agent relation within NPOs, similar to the case of POs. Based on that, we build our model by exploring the principal-agent framework within SEs.
3 The Model

The ‘earned income’ school of thought (Skloot, 1983), which assumes that the SE has ‘earned-income strategies’ exploring commercial activities in the mission realisation (Defourny and Nyssens, 2010), is the starting point for building our model. Business activities are undertaken by the SE to gain profit in order to support its social mission. Earned income strategies imply that SEs have opportunities to generate income from their individual programs (Boschee and McClurg, 2003). This approach results in a partial independence from grants and subsidies (Hoogendoorn et al., 2010). Furthermore, the nature of SEs’ activities is connected with the realisation of their social missions (Defourny and Nyssens, 2010). There are many possible objectives which SEs can pursue.

We assume that the SE delivers a social output in the quantity ($q_s$; which can be the production of a social good or service, but also an immaterial objective), at prices ($p_s$; which can be zero) and a commercial product ($q_c$), at a given price ($p_c$), on a competitive market. Both types of outputs require two factors of production: capital and labour (stock of which are $K$ and $L$ respectively), which are split to pursue the social objective ($q_s(K_s, L_s)$) and the commercial production ($q_c(K - K_s, L - L_s)$) (for a comparable model see Schiff and Weisbrod (1991) who design a model of nonprofit firms).

The capital of the SE is composed of equity ($K_E$) and debt ($K_D$) so that $K = K_E + K_D$. $K_E$ contains donated and subsidised equity ($K_{DS}$), the cost of which is zero, and owners’ equity ($K_O$), the cost of which is between zero and the cost of debt ($r_D \in (0, r)$): $K_E = K_{DS} + K_O$. Apart from donated and subsidised equity, $K_{DS} = D + G$, the organisation receives operational subsidies ($G$; from government) and operational donations ($D$; from private and individuals), which we will assume to be monetary and these are entered into the organisation’s financial result.

Depending on the allocation of resources, we can adopt this model to SEs which deliver only social objectives (when $q_c = 0$) and to SEs which provide both outputs (when $q_c \neq 0$ & $q_c \neq 0$). If $q_s$ would be zero, there would be no point in calling the enterprise ‘social’. Based on that, the model can be analysed with respect to two possible cases:

Case 1: All resources, $K$ and $L$, are exploited for a social objective, $q_c = 0$.

Case 2: All resources, $K$ and $L$, are divided between a social objective and a commercial
The first case may result in a full dependence on the subsidies and donations when \( p_s \) is close to 0. The SE does not provide any commercial activities to support the realisation of the social mission. This case can be identified in NPOs. The second one describes the SE which combines both objectives. The mission realisation is supported by the social objective and by the commercial production at the same time. In this case, the SE, which deals with a resources mix and is partly independent from subsidies and donations, can be analysed.

### 3.1 No agency problem

Owners of the SE are the group of stakeholders who invest in the SE and want the organisation to utilise their contribution for its social mission’s realisation. They are interested in activities on critical social issues, e.g. structural unemployment, social pathologies, different types of disabilities, permanent poverty, etc. For them the optimal situation is the case where the organisation exclusively focuses on the realisation of its social mission. Then the SE is free from agency problems.

In this case, the goal of the SE is the same as the aim of its owners and the problem of the SE can be formulated as follows:

\[
U^{SE} = U^O = f(q_s) \tag{P.1}
\]

s.t. \( \pi \geq 0 \)

where:

\[f(q_s)\] — social mission function of the SE.

\[\pi\] — monetary profit (corporate tax rate \( r_t \), reduced by the tax exemption rate for SEs \( \phi_S; \phi_S \leq r_t \)): the outcome of revenues from commercial sales \( (p_c q_c) \) and from social production \( (p_s q_s) \), costs of commercial production \( (c_c) \) and of social production \( (c_s) \), subsidies \( (G) \) and donations \( (D) \), reduced by the required dividend level \( (r_O K_O) \):

\[
\pi = (p_c q_c + p_s q_s - c_c - c_s + G + D)(1 - r_t + \phi_S) - r_O K_O.
\]
Implementing this approach, we analyse an organisation which in fact faces a ‘double bottom line’ - that is, an SE (Battilana and Dorado, 2010; Cornée and Szafarz, 2013). The process of the social mission realisation is supported by the sufficient level of profit, \( \pi \geq 0 \). We propose to measure social mission realisation of the SE using a function \( f \) which transforms its social realisations \( q_s \). \( q_s \) is presented as a generic variable, but it can be made more specific when the analysis is confined to a specific industry or activity. \( L_s \) and \( K_s \) (or some of their components) are the two decision variables which determine the direction of social objectives. We assume that the other variables are given and the monetary profit satisfies the constraint.

### 3.2 ‘Managerial’ Social Enterprises

The SE struggles with an agency problem when the objectives of owners do not fully overlap with the objectives of managers. Analysing the aims of managers in the SE, we try to measure how they deal with finding a balance between profit and social objectives. As mentioned by Alter (2000) and analysed by Besley and Ghatak (2013), this is the main and the overriding challenge of management in this kind of organisations.

When the SE’s policy is determined solely by its managers (severe agency problem), the problem of the SE is:

\[
U^{SE} = U^M = \pi + h(q_s) \tag{P.2}
\]

s.t. \( \pi \geq 0 \)

where:

\( h(q_s) \) – social mission function perceived by managers.

In this case, the problem of the SE \( U^{SE}; \text{P.2} \) is exactly the problem of its managers. The social objectives are realised based on the perception for managers of the social mission function \( h \). Furthermore, managers try to balance these social goals with monetary profit. The SE is run with respect to the constraint that the required level of profit should be at least equal to zero. \( L_s \) and \( K_s \) are the two decision variables which determine the direction of commercial production and of the social objective. We assume that other variables are given.
3.3 Principal - Agent compromise

We singled out two groups of stakeholders: owners and managers, each with different preferences. Therefore, the general model is built as the result of a ‘compromise’ between owners and managers.

In this framework, the principal is responsible for designing a contract which is proposed to the agent and defines the level of a yearly dividends, if required. This relationship determines obligations and benefits for both parties. The principal is obliged to pay the salary to the agent (higher than 0 for the paid managers and equal to 0 for the unpaid managers) and the agent carries out his/her duties with the aim of realising the social mission and of running a sufficiently profitable business. It should be noted that the mission realisation depends on the effort of the agent but also on random components like the general economic situation, the amount of luck, climatic conditions, etc. (Macho-Stadler and Perez-Castrillo, 2001). We assume that in the SE both parties have access to the same information and that effort is observed. Therefore, salaries can be fixed. Furthermore, in the case of the SE, a symmetric information contract can be considered. In this type of deal it is possible to verify all relevant information (Macho-Stadler and Perez-Castrillo, 2001). The contract between both parties includes both the effort which is demanded by the principal and the salary which is paid to managers. Depending on the relationship between the principal and the agent, it is possible to choose the best strategy which will contribute to the most efficient process of mission realisation. Therefore, in this model all types of stakeholders can be involved in the activities on critical social issues which are undertaken by SEs.

The general problem of the SE is defined as a maximand in which both parties influence the enterprise’s goals:

\[
\max_{L_s,K_s} \zeta U^O + (1 - \zeta)U^M, \tag{P.3}
\]

s.t. \( \pi \geq 0 \)

where:

\( \zeta \) — the bargaining power of owners versus managers, \( \zeta \in [0, 1] \), where a low value of the parameter \( \zeta \) reflects severe agency problems (Jegers, 2011a). If \( \zeta = 1 \) then the
problem P.3 reduces to the form P.1 (no agency problem); if $\zeta = 0$ then the problem P.3 reduces to the form P.2 (high agency problem),

$$U^O = \text{the owners’ utility function } U^O = f(q_s); \text{ (see: P.1)},$$

$$U^M = \text{the managers’ utility function } U^M = \pi + h(q_s); \text{ (see: P.2)},$$

Exploring this general model, we can analyse the SE which indeed is a hybrid organisation. It merges objectives of both stakeholders and mirrors the main features of SEs. The objective function which is employed (P.3) allows us to focus on an SE’s choice between fulfilling the objective of the principal and the goal of the agent. $L_s$ and $K_s$ (or some of their components) are again our decision variables which determine the direction of the social objective and of the commercial production.

4 Model application

Let us take as an example an SE which delivers only social output, at prices $p_s$ ($p_s > 0$). This output can be described by a vector $q_s = (L^S_P, q)$, where $q_s$ is hold. In this case, all the production components are involved in social activities ($L = L_s$ and $K = K_s$). The social mission of this organisation ($f(q_s)$) is a combination of the amount of jobs for a target group of social workers ($L^S_P$) and the social contribution ($q_s$).

Owners, depending on their individual preferences, prioritise two characteristics of the social mission realisation, using a parameter $\kappa$ ($\kappa \in (0, 1)$). Managers of this SE are interested in the level of the monetary profit and the social mission maximisations. They define their own weight to prioritise two characteristics of the mission realisation, a parameter $\gamma$ ($\gamma \in (0, 1)$). Furthermore, $L_s$ is composed of a social paid labour ($L^S_P$) and other (coordinating, regular and volunteers) paid and unpaid workers ($L_O$).

Applying the model for the above introduced SE, we would like to point out the optimal levels of $L^S_P$ and $K_D$ as components of total $L$ and total $K$ and based on these factors we determine the level of $q_s$. In fact, the cost of social objectives are the total cost of this organisation ($C = c_s$), while revenues from social objectives are the only income from sale ($pq = p_s q_s$). We formulate presented problem as follows:
\[
\max_{L_P^S, K_D} \zeta U^O + (1 - \zeta) U^M, \\
\text{s.t.} \quad \pi \geq 0
\]
where:

\[U^O = f(q_s) = \kappa L_P^S + (1 - \kappa)q, \text{ where } \kappa \text{ is the individual owners' preference weight,}\]

\[U^M = \pi + h(q_s) = \pi + \gamma L_P^S + (1 - \gamma)q, \text{ where } \gamma \text{ is the individual manager's parameter,}\]

In this model \(L_P^S\) and \(K_D\) are the two decision variables which determine the social objectives \((q = q_s)\), while the other variables are given. The general problem is analysed subject to the constraint that the monetary profit is equal or higher than zero.

### 4.1 General analysis

The model of the SE defined in the previous section is now solved in two steps. Firstly, the optimal levels of \(L_P^S\) and \(K_D\) are determined. Then, using the optimal values for \(L_P^S\) and \(K_D\), the optimal level of \(q\) is calculated. The Lagrangian function (4.2) for the SE’s problem takes the following form (Kuhn-Tucker):

\[
L = L_P^S(\zeta \kappa + (1 - \zeta) \gamma) + q(\zeta(1 - \kappa) + (1 - \zeta)(1 - \gamma)) + \\
(1 - \zeta) \pi + \lambda(\pi)
\]  
\tag{4.2}

The first order conditions are (4.3 - 4.5):

\[
\frac{\partial L}{\partial L_P^S} = (\zeta \kappa + (1 - \zeta) \gamma) + \frac{\partial q}{\partial L_P^S}(\zeta(1 - \kappa) + (1 - \zeta)(1 - \gamma)) + \\
+ (1 - \zeta)(1 - r_t + \phi_S) \left( p \frac{\partial q}{\partial L_P^S} - \frac{\partial C}{\partial L_P^S} \right) + \\
+ \lambda(1 - r_t + \phi_S) \left( p \frac{\partial q}{\partial L_P^S} - \frac{\partial C}{\partial L_P^S} \right) = 0
\]  
\tag{4.3}

\[1\text{ A table with the list of variables can be found in Appendix I}\]
\[
\frac{\partial L}{\partial K_D} = \frac{\partial q}{\partial K_D} (\zeta (1 - \kappa) + (1 - \zeta)(1 - \gamma)) +
(1 - \zeta)(1 - r_t + \phi_S) \left( p \frac{\partial q}{\partial K_D} - \frac{\partial C}{\partial K_D} \right) +
\lambda (1 - r_t + \phi_S) \left( p \frac{\partial q}{\partial K_D} - \frac{\partial C}{\partial K_D} \right) = 0 \quad (4.4)
\]

\[
\frac{\partial L}{\partial \lambda} = (1 - r_t - \phi_S)(pq - C + D + G) - r_O K_O = 0 \text{ or } \lambda = 0 \quad (4.5)
\]

We take into consideration two cases: the unconstrained and that of mission realisation. The combination, in which both variables are higher then zero: \( L_P^S > 0 \) and \( K_D > 0 \), is presented below (the borderline combinations, when at least one of the variables is zero, are analysed in Appendix II).

**Case 1: The unconstrained case**

Firstly, we examine the case in which the constraint is slack (\( \lambda = 0 \) & \( \pi > 0 \)). When the monetary profit is higher than 0, than the multiplier takes the form (4.6):

\[
\lambda = \frac{\frac{\partial q}{\partial K_D} (\zeta (1 - \kappa) + (1 - \zeta)(1 - \gamma)) + (1 - \zeta)(1 - r_t + \phi_S) \left( p \frac{\partial q}{\partial K_D} - \frac{\partial C}{\partial K_D} \right)}{(1 - r_t + \phi_S) \left( \frac{\partial C}{\partial K_D} - p \frac{\partial q}{\partial K_D} \right)} \quad (4.6)
\]

We can analyse this case only when the shadow price of utility (\( \lambda \)) is equal to 0. Then, the numerator of equation (4.6) is 0 while its denominator is different from 0. Defining \( X = \frac{K_D + K_E}{L_P^S + L_O} \), we get the Proposition 1.

**Proposition 1.** 2 In the unconstrained case the SE’s problem has exactly one optimal solution, \( X^* \), which is represented by a continuum of pairs \( (L_P^S, K_D) \) satisfying:

\[
X^* = \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{\mu_S}{r} - \frac{\zeta \kappa + (1-\zeta)\gamma}{(1-\zeta)(1-r_t+\phi_S)r} \right).
\]

We find, that there are a continuum of optimal combinations of labour and capital described by \( X^* \) if the overall optimum is found, \( \pi > 0 \). One may interpret the second part of the right hand side of the equation \( \left( \frac{\zeta \kappa + (1-\zeta)\gamma}{(1-\zeta)(1-r_t+\phi_S)r} \right) \) as the component influenced by the social mission. It causes social paid labour to increase in the relation to capital.

\[\text{2The Proofs of all the Propositions are contained in Appendix III}\]
Case 2: The mission realisation (emphasis) case

In a mission realisation case we consider the stage in which the profit after tax equals exactly the required yearly dividend \( \pi = 0 \). The constraint of the SE problem is binding, so now the shadow price of utility must differ from 0 \( \lambda \neq 0 \). As in the unconstrained case, we obtain that there is exactly one value of the relation between total capital and total labour which solves this problem.

**Proposition 2.** If the constraint is binding then the SE’s problem has exactly one solution for \( X^* \).

\[
\lambda \neq 0 \Rightarrow \exists ! X^* = \frac{K^*}{L^*} = \frac{K_D + K_E}{L_P^* + L_O}
\]

Based on this finding, we point out that there is exactly one value of \( X^* \) which enables us to solve the maximisation problem when the monetary profit is equal to 0. This value reflects a continuum of pairs of \( K \) and \( L \) which satisfy this criterion. One may note, however, that this arrangement might be favoured by owners because their expectations about the level of profits are satisfied and the level of the SE’s objective function is maximised.

**Proposition 3.** The optimal values of social paid labour, debt and social objective in the mission emphasis case, when the profit constraint is binding, are:

\[
L^*_{SP} = \frac{C_O - D - G - L_O(p(X^*)^\alpha - rX^*) - rK_E + \frac{\tau O K_O}{1 - \tau_1 + \phi S}}{p(X^*)^\alpha - rX^* - w_S}
\]

\[
K_D^* = (L^*_{SP} + L_O)X^* - K_E
\]

\[
q^* = (K_D^* + K_E)^\alpha (L^*_{SP} + L_O)^{(1-\alpha)}
\]

With Proposition 3 we define the levels of endogenous variables. We find, that all of them are sensitive to the cost of debt, the social labour wage, the tax rate, the tax exemption rate and all individual parameters (owners’, managers’ and bargaining power). In the subsection below we have listed, the results of the analyses of the impact of individual parameters’ values and the cost of debt.
4.2 Impact of parameters’ values

This section is divided along the same lines as the previous one. First, we analyse the unconstrained case and take into consideration the partial derivatives to present the direction of changes in the optimal values. In the second case, that of mission realisation, we consider the unique solution for the SE’s problem. In this case, when the profit after tax exactly is equal to the level which is defined in the contract, we provide the results of a numerical analysis because the partial derivatives do not give us unequivocal conclusions about the influences of the changes in parameters.

In both cases, we examine the influence of the individual parties’ preferences ($\kappa$ & $\gamma$), the bargaining power ($\zeta$) and the cost of debt ($r$).

Case 1: The unconstrained case

Based on the partial derivatives (see Appendix IV), we present the direction of changes in the optimal values in Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\kappa$</th>
<th>$\gamma$</th>
<th>$\zeta$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_P^S$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$K_D$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$q$ for $\alpha &lt; 0.5$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$q$ for $\alpha &gt; 0.5$</td>
<td>-</td>
<td>-</td>
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<td>-</td>
</tr>
</tbody>
</table>

An increase in all of the analysed parameters ($\kappa$, $\gamma$, $\zeta$ and $r$) results in a rise of $L_P^S$ and in a decrease in $K_D$. We can note that the influence on the level of the social objective is not obvious and depends on the elasticity of capital $\alpha$. A value higher than 0.5 means that the social impact drops, while a value below 0.5 results in its growth. This finding is very logical. If $\alpha$ is higher then 0.5 then capital has more influence on the social contribution. Moreover, the direction of changes in social objectives follows the direction of changes in debt.
Case 2: The mission realisation case

Based on a numerical analysis (see Appendix V), we establish, in Table 2, the relations between parameters and decision variables.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>κ</th>
<th>γ</th>
<th>ζ</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_p^S$</td>
<td>+</td>
<td>+</td>
<td>(+)</td>
<td>+</td>
</tr>
<tr>
<td>$K_D$</td>
<td>+</td>
<td>+</td>
<td>(+)</td>
<td>-</td>
</tr>
<tr>
<td>q</td>
<td>+</td>
<td>+</td>
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</tbody>
</table>

We obtain a simultaneous impact on social paid labour, debt and social output for increasing values of $κ$ and $γ$. In the case of the bargaining power $ζ$, the direction of the changes depends on the relation between the preferences of the individual parties. If $κ$ is higher (lower) then $γ$ then next to the growth of $ζ$ the amount of social paid workers, debt and production size rise (drop). When $κ$ is higher than $γ$, then owners put more attention into the creation of jobs for target people. At the same time, an increase in $ζ$ means that the influence of owners on the SE’s objective rises as well. This is exactly the reason why the number of social paid workers goes up with an increasing value of $ζ$, next to the high value of $κ$.

In both presented above cases we can note the following tendencies:

a) An increase in $κ$ leads to higher $L_p^S$. The parameter $κ$ determines individual preferences of owners about the importance of $L_p^S$ versus $q$.

b) A rise in $γ$ leads to higher stock of $L_p^S$. The strength of this impact depends on the relation between $κ$ and $γ$ but nevertheless is positive. Based on the value of $γ$, managers realise the social mission, in $U^M$. It defines their preferences concerning the creation of new jobs versus social impact. Thus, the higher $γ$ is, the higher the number of social paid labour. Moreover, when other parameters do not change, debt and social contribution increase simultaneously.

c) The cost of debt has a direct influence on the optimal size of the social impact and a

\[\text{We do not claim that they are universally valid.}\]
parallel effect of the optimal number of social paid labour. A higher (lower) level of the
cost of debt \((r)\) has an influence on the drop (rise) of debt. In practice, with a rise in the
cost of debt, the SE, as well as a PO, prefers to replace expensive capital by labour. This
supports its business activities and makes them more profitable. Our analysis shows that
higher costs of debt lead to a decline in the optimal level of debt. The proposed model
does not pay any attention to the cost of capital, but it is vital to emphasise that the cost
of debt has a significant impact on the SE’s mission realisation.

4.3 The optimal solution

In this section we compare the possible values of the SE’s utility and look for the conditions
which make that one case is preferred over the other. We use an index 1 for values from
the case 1 \((\pi > 0)\) and an index 2 for values from the case 2 \((\pi = 0)\). The SE’s problem
is presented in Figure 1.

![Figure 1: Optimal solution.](image)

We start the analysis with fixing the maximum value of profit. Calculating the first
order conditions for profit\(^4\) with respect to \(L_P^S\) and \(K_D\), we find that the maximum profit
is obtained when (4.7) applies.

\[
\frac{K_D + K_E}{L_P^S + L_O} = \frac{w_S}{r} \left( \frac{\alpha}{1 - \alpha} \right)
\]

From this condition we obtain \( K_D(*) \) as a function of \( L_P^S \). This function defines the relation between production factors for all kinds of enterprises whose main goal is to maximise a profit. In the case of SEs, the maximum profit which can be achieved is described by \( K_D(1) \), which is taken from the previous analysis \( (K_D)^5 \). Comparing \( K_D(*) \) and \( K_D(1) \) we know that the latter function \( (K_D(1)) \) has a lower slope than the former \( (K_D(*)) \). This is the result of the principal-agent relation which is adopted in the analysed model and the process of the mission realisation.

Profit after tax isoquants (such as: \( r_O K_O, \pi_C(1), \pi_C^{SE}(max) \)) are depicted in Figure 1. These functions are u-shaped because an increase in social contribution costs (one of the factors \( (L_P^S, K_D) \)) is linear whereas an increase in the social impact size increases slower than linearly (we assume a Cobb-Douglas social objectives function).

Utility isoquants have a hyperbolic shape with a negative slope. This hyperbolic shape results from the marginal rate of substitution which decreases with the further increase in one of social objectives factors. The negative slope is the consequence of the relation between social effect factors and their mutual changes \( (\frac{\partial K_D}{\partial L_P^S} < 0) \).

In case 2 when \( \pi_C = r_O K_O \) (\( \pi = 0 \)) we consider point \( B \) as the optimal solution with the utility level \( U(2) \). This borderline case is the first acceptable result which satisfies the stakeholders of the SE but it does not provide the highest value of the SE’s objective function. Nevertheless, all capital providers are satisfied because they receive their yearly dividend and the SE meets some social mission level.

Furthermore, the solution of the case 1 exists only if \( \pi_C > r_O K_O \) (\( \pi > 0 \)). All

\[
\frac{\partial \pi}{\partial L_P^S} = \left( p(1 - \alpha) \left( \frac{K_D + K_E}{L_P^S + L_O} \right)^\alpha - w_S \right) (1 - r_t + \phi_S) = 0
\]

\[
\frac{\partial \pi}{\partial K_D} = \left( pa \left( \frac{K_D + K_E}{L_P^S + L_O} \right)^{\alpha - 1} - r \right) (1 - r_t + \phi_S) = 0
\]

\[5\] A.4 Proof of Proposition 4 in Appendix III

\[6\]

\[
\frac{\partial K_D}{\partial L_P^S} = \left( - \frac{\partial U^{SE}}{\partial L_P^S} : \frac{\partial U^{SE}}{\partial K_D} \right) < 0
\]
combinations of \( L^S_p \) and \( K_D \) on the line segment between points \( C_1 \) and \( C_2 \) constitute the set of acceptable solutions. Among these solutions we can point out the one which maximises the profit for the given set of parameters that is the point \( A_3 \), with the utility value \( U_3(1) \). Points \( A_1 \) and \( A_2 \) have the same level of profit after tax (\( \pi_C(1) \)) but they differ in the value of the utility function. We get that \( A_2 \) is preferred by the SE over \( A_1 \) because with the same profit level the value of its utility is higher, \( U_2(1) > U_1(1) \).

The point \( \tilde{C} \in (C_1, C_2) \) such that \( \tilde{C} \) is the closest point to \( C_2 \) will result in the maximum value of the SE’s utility function. At this stage, profit after tax is slightly higher than in case 2 but the number of social paid labour and social objectives will be much higher. In this situation we find that the SE obtains the highest level of its objective function when its profit covers the owners’ yearly dividend and all the surpluses are reinvested in the social mission realisation (as mentioned by Dart (2004)). The SE would be able to obtain this state if it is well managed, schedules its activities and makes predictions about its financial results. However, if the access to information is not perfect or is limited and there is a moral hazard problem, then the SE might misperceive the wrong point as being the optimal choice.

5 Discussion and concluding remarks

Nowadays SEs play a crucial role in the creation of social welfare. In this paper we focus on the understanding of the mission and activities of these organisations. We propose a general principal-agent model of social organisations which describes the SE as a hybrid form. The hybridisation takes place on the individual level of donors and managers, and then is adopted to the SE as itself.

The maximisation problem of the SE is defined as a weighted average of the utilities of two groups of stakeholders who have an influence on the degree of the realisation of the mission and business orientations. On the one hand, we consider owners (the principals) who are interested in the realisation of the social mission. On the other hand, we single out managers (the agents) who try to run a profitable business and to deliver on a contract.

The proposed model enables us to analyse the process of mission realisation by taking into consideration the market environment. The SE is assumed to operate in a compet-
itive market where it can sell commercial goods or services and deliver social objectives. Furthermore, the SE can obtain benefits, such as tax exemption rates, and enjoy capital and labour that are free of cost (donations, subsidies, volunteers). However, the rate of return on the owners’ capital, which is lower than in POs with the same activities, is a typical feature of the SE.

Applying this model we can analyse internal organisation problems, such as managerial concerns, next to the nature of SEs. The model uses the parameters of the individual stakeholders as well as the bargaining power. We find that changes in any of individual parameters of parties contribute to changes in values of the determined variables and have a tangible influence on the final results. Additionally, it is possible that the principal and the agent define the process of the mission realisation in different ways. Exploiting a principal-agent approach we can concentrate on cases where one of the parties has more power than the other. Depending on the relation between individual preferences of the principal and the agent we find that the direction of changes can be opposite to the increasing or decreasing of the bargaining power. We solve the SE’s problem for a work integration SE, which delivers social objectives, in two scenarios by taking into consideration both the unconstrained case and that of the mission realisation. We point out the direction which this kind of SEs should follow to obtain the highest value of their objective functions. The desired state for SEs should be that in which SEs reinvest all surpluses in the process of their mission realisation and their profit is slightly higher than the level required by the principal’s contract.

This model is proposed with the aim of providing a useful tool for practitioners and researchers. It can be explored by both to indicate the optimal resources allocation within the field of SEs and to define the optimal state of social and business activities as well. However, we recognise some risks which may upset the fluent implementation of this solution. The expending commercialisation and bad intentions of ‘fake SEs’ may be one of the main issues which contribute to this. On the one hand, the SE may be run with a hidden commercial goal in order to explore benefits reserved for SEs. Among the others, this ulterior motive will result in the blackout the comparison without other social entities. On the other hand, it can also result in the disturbance caused by stakeholders when one of the parties has less than honourable intentions. Then, the proposed compromise will
exist only on paper and it will not be possible to implement.

We use several simplifications which can be explored in future research. First of all, we use one of the basic production function which allows us to simplify calculations and solutions. Secondly, we assume that the effort of the agent’s works is observed. Therefore, salaries are put at fixed level. In the case when the effort is not observed there is a moral hazard problem in the SE. Further, wages and the cost of debt are fixed in this model. Because of that, the increase in the amount of debt does not result in an increase in the cost of debt. Furthermore, the model is static. In a dynamic framework we can keep the surpluses ($\pi > 0$) and add them to the equity, which can be reinvested in social contribution in later years. Finally, in this paper, we focus only on two groups of stakeholders. Including more of them could be an interesting modification in further research.

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Table A.I.1: Variable definitions.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Formula or Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total income from sales</td>
<td>( pq = p_c q_c + p_s q_s )</td>
</tr>
<tr>
<td>The production function</td>
<td>( q = q_c + q_s )</td>
</tr>
<tr>
<td>Subsidies</td>
<td>( S )</td>
</tr>
<tr>
<td>Donations</td>
<td>( D )</td>
</tr>
<tr>
<td>Equity</td>
<td>( K_E )</td>
</tr>
<tr>
<td>Owners’ equity</td>
<td>( K_O )</td>
</tr>
<tr>
<td>Donated and subsidised equity</td>
<td>( K_{DS} )</td>
</tr>
<tr>
<td>Debt</td>
<td>( K_D )</td>
</tr>
<tr>
<td>Equity + liabilities = Capital</td>
<td>( K )</td>
</tr>
<tr>
<td>Total costs of activities (commercial &amp; social)</td>
<td>( C = c_c + c_s )</td>
</tr>
<tr>
<td>The number of social paid labour</td>
<td>( L_{SP} )</td>
</tr>
<tr>
<td>The number of coordinating, regular and volunteers labour</td>
<td>( L_O )</td>
</tr>
<tr>
<td>The cost of debt</td>
<td>( r )</td>
</tr>
<tr>
<td>The social labour wage</td>
<td>( w_S )</td>
</tr>
<tr>
<td>The tax rate</td>
<td>( r_t )</td>
</tr>
<tr>
<td>The rate of return on the owners’ capital</td>
<td>( r_O )</td>
</tr>
<tr>
<td>The tax exemption rate for social enterprises</td>
<td>( \phi_S )</td>
</tr>
<tr>
<td>The monetary profit</td>
<td>( \pi )</td>
</tr>
<tr>
<td>The bargaining power of owners versus managers</td>
<td>( \zeta )</td>
</tr>
<tr>
<td>The owners’ individual preferences weight</td>
<td>( \kappa )</td>
</tr>
<tr>
<td>The managers’ individual preferences weight</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>The relation between capital and labour</td>
<td>( X )</td>
</tr>
</tbody>
</table>
Appendix II

We consider four cases:

1) \( \frac{\partial L}{\partial L_p} < 0; \frac{\partial L}{\partial K_D} < 0 \)

\[
U^*_{(K_D=0; L_p=0)} = K_D^{\alpha}L_0^{1-\alpha}(\zeta(1-\kappa) + (1-\zeta)(1-\gamma)) + (1-\zeta)(1-r_t + \phi_S)(pK_E^{\alpha}L_0^{1-\alpha} - C_O + D + G) \quad (A.II.1)
\]

And:

- if \( \pi > 0 \) (\( \lambda = 0 \))
  
  Then: \( (1-r_t + \phi_S)(pK_E^{\alpha}L_0^{1-\alpha} - C_O + D + G) > r_OK_O \)

- if \( \pi = 0 \) (\( \lambda \neq 0 \))
  
  Then: \( (1-r_t + \phi_S)(pK_E^{\alpha}L_0^{1-\alpha} - C_O + D + G) = r_OK_O \)

2) \( \frac{\partial L}{\partial L_p} < 0; \frac{\partial L}{\partial K_D} > 0 \)

- if \( \pi > 0 \) (\( \lambda = 0 \))

\[
K_D \neq \left( \frac{r}{p\alpha} \right)^\frac{1}{\alpha-1} L_0 - K_E \quad (A.II.2)
\]

\[
K_D = \left( \frac{r(1-\zeta)(1-r_t + \phi_S)}{\alpha(\zeta(1-\kappa) + (1-\zeta)(1-\gamma) + (1-\zeta)(1-r_t + \phi_S)p)} \right)^\frac{1}{\alpha-1} L_0 - K_E \quad (A.II.3)
\]

Where:

- \( i \) \( \left( \frac{r(1-\zeta)(1-r_t + \phi_S)}{\alpha(\zeta(1-\kappa) + (1-\zeta)(1-\gamma) + (1-\zeta)(1-r_t + \phi_S)p)} \right) > 0 \)

- \( ii \) \( \alpha(\zeta(1-\kappa) + (1-\zeta)(1-\gamma) + (1-\zeta)(1-r_t + \phi_S)p) \neq 0 \)

- \( iii \) \( \left( \frac{r(1-\zeta)(1-r_t + \phi_S)}{\alpha(\zeta(1-\kappa) + (1-\zeta)(1-\gamma) + (1-\zeta)(1-r_t + \phi_S)p)} \right)^\frac{1}{\alpha-1} L_0 > K_E \)

Putting equations (A.II.2) and (A.II.3) together we obtain:

\[
(1-\gamma) - \zeta(\kappa-\gamma) \neq 0
\]
Then, substituting \( a = \zeta(1 - \kappa) + (1 - \zeta)(1 - \gamma) \):

\[
\left( \frac{K_D + K_E}{L_O} \right)^{\alpha} > \frac{r(1-a) - \alpha(ws + p(1-a))\left(\frac{K_D + K_E}{L_O}\right)^{\alpha - 1}}{-(1-a)ra}
\]

and:

\[
\left( \frac{K_D + K_E}{L_O} \right)^{\alpha} > \frac{rK_D + CO - D - G + \frac{rO_KO}{1 - \tau + \phi}}{pL_O}
\]

• if \( \pi = 0 \) \( (\lambda \neq 0) \)

\[
K_D \neq \left( \frac{r}{p\alpha} \right)^{\frac{1}{(\alpha - 1)}} L_O - K_E \quad \text{ (A.II.4)}
\]

\[
K_D \neq \left( \frac{r(1 - \zeta)(1 - r_t + \phi_S)}{\alpha(\zeta(1 - \kappa) + (1 - \zeta)(1 - \gamma) + (1 - \zeta)(1 - r_t + \phi_S)p)} \right)^{\frac{1}{(\alpha - 1)}} L_O - K_E \quad \text{ (A.II.5)}
\]

Then, substituting \( a = \zeta(1 - \kappa) + (1 - \zeta)(1 - \gamma) \):

\[
\left( \frac{K_D + K_E}{L_O} \right)^{(1-\alpha)} > \frac{\alpha(ws + p(1-a)) - \left(\frac{K_D + K_E}{L_O}\right)^{1-\alpha}ra}{r(1-a)}
\]

and:

\[
(1 - r_t + \phi_S)(p(K_D + K_E)^{\alpha}L_O^{(1-\alpha)} - rK_D - CO + D + G) = rO_KO
\]

For both sub-cases we have:

\[
U^*_{(K_D;L_O^\pi=0)} = (K_D + K_E)^{\alpha}L_O^{1-\alpha}(\zeta(1 - \kappa) + (1 - \zeta)(1 - \gamma)) +
\]

\[
+ (1 - \zeta)(1 - r_t + \phi_S)(p(K_D + K_E)^{\alpha}L_O^{1-\alpha} - rK_D - CO + D + G) \quad \text{ (A.II.6)}
\]

3) \( L_P^S > 0 \ & \ \frac{\partial \mathcal{L}}{\partial L_P^S} = 0; \ K_D = 0 \ & \ \frac{\partial \mathcal{L}}{\partial K_D} < 0 \)

• if \( \pi > 0 \) \( (\lambda = 0) \)

\[
L_P^S \neq \frac{K_E}{\left( \frac{ws}{p(1-a)} \right)^{\frac{1}{\pi}}} - L_O \quad \text{ (A.II.7)}
\]
\[
L_P^S = \frac{K_E}{\left( \frac{(1-\zeta)(1-r_t+\phi_S)w_S-\zeta\kappa-(1-\zeta)\gamma}{(1-\alpha)(\zeta(1-\kappa)+(1-\zeta)(1-\gamma)+(1-\zeta)(1-r_t+\phi_S)p)} \right)^{\frac{1}{\alpha}}} - L_O \quad \text{(A.II.8)}
\]

Putting equations (A.II.7) and (A.II.8) together we obtain:

\[
p \neq -\frac{w_S(\zeta(1-\kappa)+(1-\zeta)(1-\gamma))}{(\zeta+(1-\zeta)\gamma)}
\]

Further:

\[i)\left( \frac{(1-\zeta)(1-r_t+\phi_S)w_S-\zeta\kappa-(1-\zeta)\gamma}{(1-\alpha)(\zeta(1-\kappa)+(1-\zeta)(1-\gamma)+(1-\zeta)(1-r_t+\phi_S)p)} \right) \neq 0\]

\[ii) \quad ((1-\zeta)(1-r_t+\phi_S)w_S-\zeta\kappa-(1-\zeta)\gamma) > 0\]

\[iii) \quad \frac{K_E}{\left( \frac{(1-\zeta)(1-r_t+\phi_S)w_S-\zeta\kappa-(1-\zeta)\gamma}{(1-\alpha)(\zeta(1-\kappa)+(1-\zeta)(1-\gamma)+(1-\zeta)(1-r_t+\phi_S)p)} \right)^{\frac{1}{\alpha}}} > L_O\]

Then, substituting \(a = \zeta(1-\kappa)+(1-\zeta)(1-\gamma)\):

\[
\left( \frac{K_E}{L_P^S+L_O} \right)^{\frac{\alpha}{\left( \frac{K_E}{L_P^S+L_O} \right)^{\frac{1}{\alpha}}}} > \frac{r(1-a)-a(w_S a+p(1-a))}{-(1-a)a} \left( \frac{K_E}{L_P^S+L_O} \right)^{\frac{1-a}{\alpha}}
\]

and:

\[
\left( \frac{K_E}{L_P^S+L_O} \right)^{\frac{\alpha}{\left( \frac{K_E}{L_P^S+L_O} \right)^{\frac{1}{\alpha}}}} > \frac{w_S L_P^S+C_O-D-G+\frac{\alpha K_O}{1-r_t+\phi_S}}{p(L_P^S+L_O)}
\]

• if \(\pi = 0 \quad (\lambda \neq 0)\)

\[
L_P^S \neq \frac{K_E}{\left( \frac{w_S}{p(1-\alpha)} \right)^{\frac{1}{\alpha}}} - L_O \quad \text{(A.II.9)}
\]

\[
L_P^S \neq \frac{K_E}{\left( \frac{(1-\zeta)(1-r_t+\phi_S)w_S-\zeta\kappa-(1-\zeta)\gamma}{(1-\alpha)(\zeta(1-\kappa)+(1-\zeta)(1-\gamma)+(1-\zeta)(1-r_t+\phi_S)p)} \right)^{\frac{1}{\alpha}}} - L_O \quad \text{(A.II.10)}
\]

Then, substituting \(a = \zeta(1-\kappa)+(1-\zeta)(1-\gamma)\):

\[
\left( \frac{K_E}{L_P^S+L_O} \right)^{(1-a)} > \frac{a(w_S a+p(1-a))-r(1-a)a\left( \frac{K_E}{L_P^S+L_O} \right)}{r(1-a)}
\]

and:

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\[(1 - r_t + \phi_S)(p(K_E)\alpha(L_P^S + L_O)^{(1-\alpha)} - w_SL_P^S - C_O + D + G) = r_OK_O\]

For both sub-cases we have:

\[U^*_{(K_D=0;L_P^S)} = (K_E)^\alpha(L_P^S + L_O)^{1-\alpha}(\zeta(1-\kappa) + (1-\zeta)(1-\gamma)) + (1-\zeta)(1-r_t + \phi_S)(p(K_E)^\alpha(L_P^S + L_O)^{1-\alpha} - w_SL_P^S - C_O + D + G) \]

(A.II.11)

4) \[L_P^S > 0 \& \frac{\partial C}{\partial L_P^S} = 0; \quad K_D > 0 \& \frac{\partial C}{\partial K_D} = 0\]

This case is presented in the main body of the paper.
Appendix III

A.1 Proof of Proposition 1

Setting $X \equiv \frac{K_D + K_E}{L_D + L_O}$, we obtain:

$$X = \left( \frac{\alpha}{r} \left( \frac{\zeta(1 - \kappa)}{(1 - \zeta)(1 - r_t - \phi_S)} + \frac{(1 - \gamma)}{(1 - r_t - \phi_S)} \right) \right)^{\frac{1}{1 - \alpha}}, \quad (A.III.1)$$

while:

$$\left( \frac{p\alpha}{r} \right)^{\frac{1}{1 - \alpha}} \neq X.$$

Taking equations (4.3) and (4.4) we obtain that:

$$X = \left( \frac{\alpha}{(1 - \alpha)} \right) \left( \frac{w_S}{r} - \frac{\zeta \kappa + (1 - \zeta)\gamma}{(1 - \zeta)(1 - r_t + \phi_S)r} \right), \quad (A.III.2)$$

when the shadow price of utility equals 0.

Putting equations (A.III.1) and (A.III.2) together we get the condition that this situation occurs when:

$$\hat{p} = \left( \frac{\hat{w}_S}{1 - \alpha} \right)^{(1 - \alpha)} \left( \frac{\zeta}{\alpha} \right)^{\alpha}$$

where:

$$\hat{p} = p + \frac{(1 - \gamma)}{(1 - r_t + \phi_S)} + \frac{\zeta(1 - \kappa)}{(1 - \zeta)(1 - r_t + \phi_S)}$$

$$\hat{w}_S = w_S - \frac{\zeta \kappa + (1 - \zeta)\gamma}{(1 - \zeta)(1 - r_t + \phi_S)}$$

This extra condition gives us the relationship between the cost of hiring social paid labour and the cost of debt subject to the individual parameters of the involved parties. We obtain the price level ($\hat{p}$) which enables the SE to maximise the problem of the SE.

$$\blacksquare$$

A.2 Proof of Proposition 2

Having the first derivatives (4.3-4.5) and the equation which describes the shadow value (4.6) we can substitute $\lambda$ in the equation 4.3. We obtain an expression of the form:
\[ AX^\alpha - BX^{(\alpha - 1)} + C = 0 \]

Multiplying by \( X^{(1-\alpha)} \):

\[ f(X) = AX - B + CX^{(1-\alpha)} \]

Where:

\[
A = (1 - \alpha)(\zeta(1 - \kappa) + (1 - \zeta)(1 - \gamma))r; \]

\[
B = \alpha(w_S(\zeta(1 - \kappa) + (1 - \zeta)(1 - \gamma)) + p(\zeta\kappa + (1 - \zeta)\gamma)); \]

\[
C = r(\zeta\kappa + (1 - \zeta)\gamma). \]

The equation \( f(X) \) has a solution because it is continuous, strictly monotonic and has limits of opposite sign:

\[
\lim_{x \to 0} = -B
\]

\[
\lim_{x \to \infty} = +\infty
\]

\( \forall X \geq 0 \):

\[
\frac{\partial f(X)}{\partial X} > 0
\]

\[ \blacksquare \]

**A.3 Proof of Proposition 3**

\( \lambda \neq 0 \Rightarrow \pi = 0 \)

Using the constraint which is binding (\( \pi = 0 \)) and knowing the value of \( X^* \) from the Proof of Proposition 5 we can determine the optimal values if \( L_P^S \), \( K_D \) and \( q \).

\[
L_P^S = \frac{C_0-D-G-L_0(p(X^*)^\alpha-rX^*)-rK_E+r^\alpha K_D}{p(X^*)^\alpha-rX^*-w_S}
\]

\[
K_D = (L_P^S + L_O)X^* - K_E
\]

\[
q^* = (K_D + K_E)^\alpha(L_P^S + L_O)^{(1-\alpha)}
\]

\[ \blacksquare \]
A.4 Proof of Proposition 4

We put together equation (A.III.2) from the Proof of Proposition 1 and the known value of $U^*(L^*_P, K^*_D)$ which we obtain after inserting values $L^*_P$ and $K^*_D$ from the Proof of Proposition 6. We have:

$$\left(\frac{\alpha}{(1-\alpha)}\right) \left(\frac{w_S}{r} - \frac{\zeta \kappa + \gamma(1-\zeta)}{(1-\zeta)(1-r_1 + \phi_S)r}\right)(L^*_P + L_O) - K_E = U^*(L^*_P, K^*_D)$$

Therefore:

$$\tilde{L}_P^S = \frac{U^*(L^*_P, K^*_D) + K_E}{\left(\frac{w_S}{r} - \frac{\zeta \kappa + \gamma(1-\zeta)}{(1-\zeta)(1-r_1 + \phi_S)r}\right)} - L_O$$

$$\tilde{K}_D = \left(\frac{\alpha}{(1-\alpha)}\right) \left(\frac{w_S}{r} - \frac{\zeta \kappa + \gamma(1-\zeta)}{(1-\zeta)(1-r_1 + \phi_S)r}\right)(L^*_P + L_O) - K_E$$

Then:

$$U^*(L^*_P, K^*_D) = U^*(\tilde{L}_P^S, \tilde{K}_D)$$

And $\forall \epsilon > 0$:

$$U^*(L^*_P, K^*_D) < U^*(\tilde{L}_P^S + \epsilon, \tilde{K}_D)$$

$$U^*(L^*_P, K^*_D) < U^*(\tilde{L}_P^S, \tilde{K}_D + \epsilon)$$

\[\blacksquare\]
Appendix IV

From the equation (A.III.2) we obtain that:

\[ K_D = \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{w_S}{r} - \frac{\zeta \kappa + \gamma (1-\zeta)}{(1-\zeta)(1-r_l+\phi_S)r} \right) (L_P^S + L_O) - K_E \quad \text{(A.IV.1)} \]

\[ L_P^S = \frac{K_D + K_E}{\left( \frac{\alpha}{1-\alpha} \right) \left( \frac{w_S}{r} - \frac{\zeta \kappa + \gamma (1-\zeta)}{(1-\zeta)(1-r_l+\phi_S)r} \right)} - L_O \quad \text{(A.IV.2)} \]

\[ q = (K_D + K_E)^\alpha (L_P^S + L_O)^{1-\alpha} = (X(L_P^S + L_O))^\alpha \left( \frac{K_D + K_E}{X} \right)^{1-\alpha} \quad \text{(A.IV.3)} \]

Then, we calculate partial derivatives with respect to \( \kappa \):

\[ \frac{\partial K_D}{\partial \kappa} = - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{1}{(1-r_l+\phi_S)r} \right) (L_P^S + L_O) < 0 \]

\[ \frac{\partial L_P^S}{\partial \kappa} = - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{K_D + K_E}{\left( \frac{w_S}{r} - \frac{\zeta \kappa + \gamma (1-\zeta)}{(1-\zeta)(1-r_l+\phi_S)r} \right)} \right) \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{1}{(1-r_l+\phi_S)r} \right) > 0 \]

\[ \frac{\partial q}{\partial \kappa} = \alpha(X(L_P^S + L_O))^{(\alpha-1)} \frac{\partial K_D}{\partial \kappa} \left( \frac{K_D + K_E}{X} \right)^{(1-\alpha)} + \]

\[ + (X(L_P^S + L_O))^{\alpha} (1-\alpha) \left( \frac{K_D + K_E}{X} \right)^{-\alpha} \left( \frac{\partial L_P^S}{\partial \kappa} \right) > 0 \Leftrightarrow 1 < \frac{\alpha}{1-\alpha} \Rightarrow \alpha > 0.5 \]

We calculate partial derivatives with respect to \( \gamma \) of equations (A.IV.1, A.IV.2 and A.IV.3)

\[ \frac{\partial K_D}{\partial \gamma} = - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{1}{(1-r_l+\phi_S)r} \right) (L_P^S + L_O) < 0 \]

\[ \frac{\partial L_P^S}{\partial \gamma} = - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{K_D + K_E}{\left( \frac{w_S}{r} - \frac{\zeta \kappa + \gamma (1-\zeta)}{(1-\zeta)(1-r_l+\phi_S)r} \right)} \right) \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{1}{(1-r_l+\phi_S)r} \right) > 0 \]

\[ \frac{\partial q}{\partial \gamma} = \alpha(X(L_P^S + L_O))^{(\alpha-1)} \frac{\partial K_D}{\partial \gamma} \left( \frac{K_D + K_E}{X} \right)^{(1-\alpha)} + \]

\[ + (X(L_P^S + L_O))^{\alpha} (1-\alpha) \left( \frac{K_D + K_E}{X} \right)^{-\alpha} \left( \frac{\partial L_P^S}{\partial \gamma} \right) > 0 \Leftrightarrow 1 < \frac{\alpha}{1-\alpha} \Rightarrow \alpha > 0.5 \]

We calculate partial derivatives with respect to \( \zeta \) of equations (A.IV.1, A.IV.2 and A.IV.3):

\[ \frac{\partial K_D}{\partial \zeta} = - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{\kappa(1-\zeta) + \zeta \kappa}{(1-\zeta)(1-r_l+\phi_S)r} \right) (L_P^S + L_O) < 0 \]

\[ \frac{\partial L_P^S}{\partial \zeta} = - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{K_D + K_E}{\left( \frac{w_S}{r} - \frac{\zeta \kappa + \gamma (1-\zeta)}{(1-\zeta)(1-r_l+\phi_S)r} \right)} \right) \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{\kappa(1-\zeta) + \zeta \kappa}{(1-\zeta)(1-r_l+\phi_S)r} \right) (L_P^S + L_O) > 0 \]
\[ \frac{\partial q}{\partial \zeta} = \alpha (X(L_P^S + L_G))^{(\alpha - 1)} \frac{\partial K_D}{\partial \zeta} \left( \frac{K_D + K_E}{X} \right)^{(1-\alpha)} + \\
+ (X(L_P^S + L_G))^\alpha (1 - \alpha) \left( \frac{K_D + K_E}{X} \right)^{-\alpha} \left( \frac{\partial L_P^S}{\partial \zeta} \right) > 0 \iff 1 < \frac{\alpha}{1-\alpha} \Rightarrow \alpha > 0.5 \]
Appendix V\textsuperscript{7}

Figure A.V.1: Changes in social paid labour with respect to $\zeta$ ($\kappa = 0.55$, $\&$ $\gamma = 0.45$ or $\kappa = 0.45$ $\&$ $\gamma = 0.55$).

Figure A.V.2: Changes in debt with respect to $\zeta$ ($\kappa = 0.55$, $\&$ $\gamma = 0.45$ or $\kappa = 0.45$ $\&$ $\gamma = 0.55$).

Figure A.V.3: Changes in social output with respect to $\zeta$ ($\kappa = 0.55$, $\&$ $\gamma = 0.45$ or $\kappa = 0.45$ $\&$ $\gamma = 0.55$).

Figure A.V.4: Changes in social paid labour and social output with respect to changes in $\gamma$ ($\kappa = 0.25$ $\&$ $\zeta = 0.5$).

Figure A.V.5: Changes in social paid labour and social output with respect to changes in $\kappa$ ($\gamma = 0.45$ $\&$ $\zeta = 0.5$).

Figure A.V.6: Changes in social paid labour and social output with respect to changes in $r$ ($\kappa = 0.25$, $\gamma = 0.5$ $\&$ $\zeta = 0.8$).

\textsuperscript{7}Used values: $\alpha = 0.3$, $p = 10$, $w_S = 5$, $r = 0.3$, $C_O - D - G + \frac{r_0 R_0}{1 - r_0 + \phi_0} = 1000$, $K_E = 0$, $L_O = 0$
References


